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# Linear Algebra and its Applications

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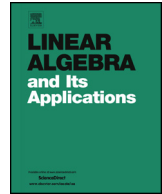
# Operators without eigenvalues in finite-dimensional vector spaces

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## A R T I C L E   I N F O

## A B S T R A C T

We introduce the concept of a canonical subspace of $\mathbb{C}^d[z]$ and among other results prove the following statements. An operator in a finite-dimensional vector space has no eigenvalues if and only if it is similar to the operator of multiplication by the independent variable on a canonical subspace of $\mathbb{C}^d[z]$. An operator in a finite-dimensional Pontryagin space is symmetric and has no eigenvalues if and only if it is isomorphic to the operator of multiplication by the independent variable in a canonical subspace of $\mathbb{C}^d[z]$ with an inner product determined by a full matrix polynomial Nevanlinna kernel.

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## 1. Introduction

Let $d \in \mathbb{N}$, the set of natural numbers. A *canonical subspace* $\mathfrak{C}$ of $\mathbb{C}^d[z]$ is a space of $d \times 1$ vector functions of the form

$$\begin{bmatrix} p_1(z) \\ \vdots \\ p_d(z) \end{bmatrix}$$

in which $p_k(z)$ is a scalar polynomial of degree $< \mu_k$, where $\mu_k \in \mathbb{N}$ and $k \in \{1, \ldots, d\}$. We collect the $\mu_k$'s in one tuple $\mu = (\mu_1, \ldots, \mu_d)$, assume throughout that they are ordered $\mu_1 \geq \cdots \geq \mu_d$ and denote the space $\mathfrak{C}$ by $\mathfrak{C}_\mu$. The operator $S_{\mathfrak{C}_\mu}$ of multiplication by the independent variable $z$ in $\mathfrak{C}_\mu$ determines a second decreasing tuple $\delta = (\delta_1, \ldots, \delta_m)$ in which $\delta_k = \dim\big(\mathrm{dom}(S_{\mathfrak{C}_\mu})^{k-1}\big)$ for $k \in \{1, \ldots, m\}$ and $m \in \mathbb{N}$ is the smallest natural number such that $\dim\big(\mathrm{dom}(S_{\mathfrak{C}_\mu})^m\big) = 0$. The relation between the tuples $\mu$ and $\delta$ can be conveniently formulated by using three kinds of operators on the set of all nonincreasing tuples: $\mathsf{Con}$, $\mathsf{Int}$ and $\mathsf{Der}$, see Section 5. Each of the tuples $\mu$, $\delta$ and a third nonincreasing tuple determine a so-called Young diagram which can be used to represent a canonical subspace of $\mathbb{C}^d[z]$.

The tuples $\mu$ and the tuple $\mathsf{Con}\,\mu$ are closely related to the Segre characteristic and Weyr characteristic of a nilpotent matrix as defined in [27]; see Remark 8.6 and Example 8.7.

In this paper we study operators $S$ without eigenvalues defined in finite-dimensional vector spaces $\mathfrak{F}$; in [26] and [7] such operators are called multishifts. We prove that such an operator $S$ is similar to the operator $S_{\mathfrak{C}_\mu}$ of multiplication by the independent variable $z$ on a canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$. Here $d = \mathrm{codim}(\mathrm{dom}\,S)$. In this sense the pair $(\mathfrak{C}_\mu, S_{\mathfrak{C}_\mu})$ serves as a model for the pair $(\mathfrak{F}, S)$. The number of such models with $\dim \mathfrak{C}_\mu = n$ is finite; in fact it equals $\mathsf{p}(n)$, the number of integer partitions of $n$, see Corollary 7.2. In the last part of the paper we study symmetric operators without eigenvalues defined in finite-dimensional vector spaces with nondegenerate inner products. (Since this research was motivated by research involving infinite-dimensional Pontryagin spaces, see [10], we will call these spaces finite-dimensional Pontryagin spaces.) We prove that such an operator is unitarily equivalent to the operator $S_{\mathfrak{C}_\mu}$ of multiplication by the independent variable in a canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$ which is equipped with a nondegenerate inner product that makes $S_{\mathfrak{C}_\mu}$ symmetric. Since $\mathfrak{C}_\mu$ is finite-dimensional, this inner product on $\mathfrak{C}_\mu$ makes $\mathfrak{C}_\mu$ a reproducing kernel Pontryagin space with a reproducing kernel determined by a $d \times 2d$ matrix polynomial $\mathscr{P}(z)$. We prove that this $\mathscr{P}(z)$ is characterized by four of its properties, see Theorem 10.4(C).

By definition a canonical space of vector polynomials is a canonical subspace of $\mathbb{C}^d[z]$ for some $d \in \mathbb{N}$. In Section 6 we show that a linear bijection between two canonical spaces of vector polynomials intertwines the operators of multiplication by the independent variable in these spaces if and only if the spaces coincide with the same $\mathfrak{C}_\mu$ and the

bijection is the operator of multiplication by a unimodular block upper triangular matrix $W(z)$; the sizes of the blocks are determined by the entries of $\mu$, see Theorem 6.2.

The operator $S = S_{\mathfrak{C}_\mu}$ of multiplication by $z$ in $\mathfrak{C}_\mu \subset \mathbb{C}^d[z]$ has the following properties:

(a) $\operatorname{codim}(\operatorname{dom} S) = d > 0$,
(b) $S$ has no eigenvalues.

We show in Theorem 7.1 that if $\mathfrak{F}$ is a finite-dimensional vector space and $S$ is an operator in $\mathfrak{F}$ satisfying (a) and (b) then there is a linear bijection $\Phi$ from $\mathfrak{F}$ onto a canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$ that intertwines $S$ with the multiplication operator $S_{\mathfrak{C}_\mu}$. Here the tuple $\mu$ is uniquely determined by the tuple $\delta$ of positive dimensions of $\operatorname{dom} S^{k-1}$ using the operators Con, Int and Der introduced in Section 5. The tuple $\mu$ is an important tool to describe a special basis for $\mathfrak{F}$ in Theorem 7.1(I). A closely related result is given in [30], [26, Theorem 8.1] and [7, Proposition 4.8].

In Section 8 we study a special nilpotent extension of the operator of multiplication by the independent variable on a canonical space of vector polynomials, its relation to the differentiation operator and to the classical Jordan decomposition of an arbitrary everywhere defined operator on a finite-dimensional vector space. The last two theorems in Section 8 establish two bijective relations between equivalence classes of operators without eigenvalues and equivalence classes of nilpotent operators, see Theorems 8.3 and 8.4.

Section 9 serves as an intermezzo between sections in which we use only the linear structure of finite-dimensional vector spaces and sections in which we consider nondegenerate inner products on finite-dimensional vector spaces and symmetric operators without eigenvalues in those spaces. We show that for an arbitrary operator $S$ without eigenvalues there exists an inner product in which $S$ is symmetric and that there are no restrictions on the signature of this inner product. We first prove this for a shift operator, see Definition 9.1, and then we use the fact that each operator without eigenvalues is a direct sum of shifts. Self-adjoint extensions of shifts in finite-dimensional Pontryagin spaces have been studied by Lander in [22].

In Section 10 we investigate canonical subspaces equipped with a Pontryagin space inner product under which the operator of multiplication by the independent variable is symmetric. Let $Q$ be a self-adjoint matrix with $d$ positive and $d$ negative eigenvalues and let $\mathscr{P}(z)$ be a $d \times 2d$ matrix polynomial such that the rank of $\mathscr{P}(z)$ is $d$ for some $z \in \mathbb{C}$ and

$$\mathscr{P}(z)Q^{-1}\mathscr{P}(z^*)^* = 0, \quad \text{for all} \quad z \in \mathbb{C}.$$

Consider the matrix polynomial Nevanlinna kernel

$$K_{\mathscr{P}}(z, w) := \frac{\mathrm{i}}{z - w^*} \mathscr{P}(z)Q^{-1}\mathscr{P}(w)^*, \quad z \neq w^*, \quad z, w \in \mathbb{C},$$

which we studied in [10] and called a full matrix polynomial Nevanlinna kernel if rank $\mathscr{P}(z) = d$ for all $z \in \mathbb{C}$, see [10, p. 1320]. In Theorem 10.3 we formulate sufficient conditions on $\mathscr{P}(z)$ which ensure that the reproducing kernel space with this reproducing kernel $K_{\mathscr{P}}(z, w)$ is the canonical subspace $\mathfrak{C}_\mu$ and the operator $S_{\mathfrak{C}_\mu}$ is symmetric. The entries of the tuple $\mu$ are the Forney indices of the matrix polynomial $\mathscr{P}(z)$, see Section 4. Conditions which ensure that the reproducing kernel space is of the form $\mathscr{W}(z)\mathfrak{C}_\mu$ for some $d \times d$ matrix polynomial $\mathscr{W}(z)$ are considered in Theorem 10.4. Parts (A) and (B) of Theorem 10.4 combined with Theorem 10.3 provide a characterization of canonical spaces of vector polynomials as reproducing kernel spaces. This characterization is given in Theorem 10.4 part (C). Here and in Section 11 we use properties of matrix polynomials collected in Section 4.

In Section 11 we construct in two ways a model $(\mathfrak{C}, S_{\mathfrak{C}})$ for the pair $(\mathfrak{G}, S)$ where $\mathfrak{G}$ is a finite-dimensional Pontryagin space and $S$ is a symmetric operator in $\mathfrak{G}$ without eigenvalues. The first construction makes use of a self-adjoint operator extension of $S$ whose existence follows from Lemma 3.5 and the second construction is based on [10, Theorem 1.1]. The inner product on the space $\mathfrak{C}$ in the model comes from a reproducing kernel determined by a polynomial with properties (a)–(d) of Theorem 10.3.

We conclude with Section 12 in which we present examples to illustrate our results.

Although most proofs in this paper are based on methods from linear algebra, in the sequel we assume that the reader is familiar with Pontryagin spaces and multi-valued operators on such spaces such as symmetric and self-adjoint relations (as in [19] and [11]) and reproducing kernel Pontryagin spaces (as in [2, Chapter 1] and [1, Chapter 7]).

## 2. Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers and complex numbers. In addition to this standard notation we use $\mathbb{N}_0$ to denote the set of all nonnegative integers.

For a finite set $F$ by $\#F$ we denote the cardinality of $F$. In case of a tuple $\mu$ the symbol $\#\mu$ denotes the length of $\mu$ and $\sum\mu$ stands for the sum of the entries in $\mu$.

With $d \in \mathbb{N}$, the vector space of all $d \times 1$ vectors with complex entries is written as $\mathbb{C}^d$. Similarly, with $m \in \mathbb{N}$, by $\mathbb{C}^{d \times m}$ we denote the space of all complex $d \times m$ matrices. $I_d$ stands for the $d \times d$ identity matrix and $Z_d$ is the $d \times d$ matrix obtained by reversing the order of the columns of $I_d$.

For $k \in \{1, \ldots, d\}$ by $e_{d,k} \in \mathbb{C}^d$ we denote the $k$-th column of $I_d$ and $E_{d,k}$ denotes the $d \times k$ matrix which embeds the space $\mathbb{C}^k$ onto the subspace of $\mathbb{C}^d$ spanned by the first $k$ columns of the identity matrix $I_d$ such that $E_{d,k}e_{k,j} = e_{d,j}$ for $j \in \{1, \ldots, k\}$.

By $\mathbb{C}^d[z]$ we denote the space of all vector polynomials with coefficients in $\mathbb{C}^d$ and by $\mathbb{C}^{d \times m}[z]$ the space of all matrix polynomials with coefficients in $\mathbb{C}^{d \times m}$. The spaces $\mathbb{C}^d$ and $\mathbb{C}^{d \times m}$ are identified with the subspaces of all constant polynomials in $\mathbb{C}^d[z]$ and $\mathbb{C}^{d \times m}[z]$, respectively.

Vector polynomials, that is members of spaces $\mathbb{C}^d[z]$ will be denoted by lower case Latin letters, sometimes with, sometimes without the variable: $f(z)$ or $f$. Matrix polynomials, that is members of spaces $\mathbb{C}^{d \times m}[z]$ will be denoted by upper case calligraphic letters, sometimes with, sometimes without the variable, $\mathscr{P}(z)$ or $\mathscr{P}$.

For a polynomial in any of these spaces of polynomials we define its *degree*. The *degree of a zero polynomial* is $-\infty$. The *degree of a nonzero polynomial* is the highest power of $z$ for which the corresponding coefficient is nonzero. For example, for a nonzero $\mathscr{P}(z) \in \mathbb{C}^{d \times m}[z]$ with

$$\mathscr{P}(z) = P_0 + P_1 z + \cdots + P_n z^n$$

we define

$$\deg \mathscr{P}(z) := \max\big\{ k \in \{0, \ldots, n\} \ : \ P_k \neq 0 \big\}.$$

If the maximum on the right-hand side of the preceding definition is $m$, then the coefficient $P_m$ is called the *leading coefficient of* $\mathscr{P}(z)$. By definition, $P_m \neq 0$.

For a set of polynomials $\mathfrak{A}$ and $n \in \mathbb{N}_0$ the symbol $\mathfrak{A}_{<n}$ stands for the set all polynomials in $\mathfrak{A}$ whose degree is strictly less than $n$. For example, $\mathbb{C}^d[z]_{<1} = \mathbb{C}^d$ and $\mathbb{C}^d[z]_{<0} = \{0\}$.

A square matrix polynomial $\mathscr{U}(z) \in \mathbb{C}^{d \times d}[z]$ is said to be *unimodular* if $\det \mathscr{U}(z)$ is a nonzero constant. The inverse of a unimodular $\mathscr{U}(z) \in \mathbb{C}^{d \times d}[z]$ belongs to $\mathbb{C}^{d \times d}[z]$ and it is also unimodular.

A $d \times d$ matrix function $K(z, w)$ is said to be a *polynomial Hermitian kernel* if it is a polynomial of two variables $z$ and $w^*$ and $K(z, w)^* = K(w, z)$ for all $z, w \in \mathbb{C}$. The last equality implies that the degree of $K(z, w)$ as a polynomial in $z$ equals the degree of $K(z, w)$ as a polynomial in $w^*$ and this common value will be called the *degree of* $K(z, w)$.

Capital letters in Fraktur alphabet $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \ldots$ will denote vector spaces over $\mathbb{C}$. Usually letters at the beginning of the alphabet will denote spaces of vector polynomials. We are primarily interested in finite-dimensional spaces.

For a subspace $\mathfrak{C}$ of $\mathbb{C}^d[z]$ by $S_{\mathfrak{C}}$ we denote the operator of multiplication by the independent variable in $\mathfrak{C}$. More precisely,

$$\operatorname{dom} S_{\mathfrak{C}} = \big\{ f(z) \in \mathfrak{C} \ : \ z f(z) \in \mathfrak{C} \big\}$$

and

$$(S_{\mathfrak{C}} f)(z) = z f(z) \quad \text{for all} \quad f \in \operatorname{dom} S_{\mathfrak{C}}.$$

In a finite-dimensional vector space $\mathfrak{F}$ we will use the letter $S$ without a subscript to denote a linear operator without eigenvalues which is defined on a proper subset of $\mathfrak{F}$ with values in $\mathfrak{F}$. The letter $A$ will be used to denote a linear operator $A : \mathfrak{F} \to \mathfrak{F}$ defined

on the entire vector space. As usual $I$ stands for the identity operator. The underlying space will always be clear from the context. In formulas involving operators we often abbreviate $zI$ to $z$ when $z \in \mathbb{C}$. For example, we write $S - z$ instead of $S - zI$.

We often identify an operator in $\mathfrak{F}$ with its graph in

$$\mathfrak{F}^2 = \big\{\{u, v\} \ : \ u, v \in \mathfrak{F}\big\}$$

and then we denote them by the same symbol. Here the notation $\{u, v\}$ stands for the ordered pair; although we use curly brackets also to denote sets, the meaning will be clear from the context. For example, the operator $\alpha I$ on $\mathfrak{F}$, $\alpha \in \mathbb{C}$, is identified with $\alpha I = \big\{\{u, \alpha u\} \ : \ u \in \mathfrak{F}\big\}$. A *linear relation* $S$ in $\mathfrak{F}$ is a linear subset $S$ of $\mathfrak{F}^2$. It is the graph of an operator $S$ if and only if $\{0, v\} \in S$ implies $v = 0$, and then we write $Su = v$ for $\{u, v\} \in S$. We treat a linear relation $S$ in $\mathfrak{F}$ as if it is an operator and define the *domain* dom $S$, the *range* ran $S$ and the *kernel* ker $S$ of $S$ by

$$\text{dom}\, S = \big\{u \ : \ \{u, v\} \in S\big\}, \ \text{ran}\, S = \big\{v \ : \ \{u, v\} \in S\big\}, \ \text{ker}\, S = \big\{u \ : \ \{u, 0\} \in S\big\}$$

and for $u \in \text{dom}\, S$ we set $S(u) = \big\{v \ : \ \{u, v\} \in S\big\}$, $S(u) = Su$ if $S$ is an operator. The linear relations

$$S^{-1} = \big\{\{v, u\} \ : \ \{u, v\} \in S\big\}, \quad S|_{\mathfrak{G}} = \big\{\{u, v\} \ : \ \{u, v\} \in S, u \in \mathfrak{G}\big\}$$

and

$$\alpha S = \big\{\{u, \alpha v\} \ : \ \{u, v\} \in S\big\}$$

are called the *inverse* of $S$, the *restriction* of $S$ to a subset $\mathfrak{G}$ of $\mathfrak{F}$ and the *product* of $S$ by $\alpha \in \mathbb{C}$. The *sum* and *difference* $S \pm T$ and the *product* $TS$ of two linear relations $S$ and $T$ in $\mathfrak{F}$ are defined by

$$S \pm T = \big\{\{u, v \pm w\} \ : \ \{u, v\} \in S, \{u, w\} \in T\big\}$$

and

$$TS = \big\{\{u, w\} \ : \ \{u, v\} \in S, \{v, w\} \in T \text{ for some } v \in \mathfrak{F}\big\}.$$

For example, $\alpha S = (\alpha I)S$ and, since $S - \alpha = S - \alpha I = \big\{\{u, v - \alpha u\} \ : \ \{u, v\} \in S\big\}$, we have $\text{ker}(S - \alpha) = \text{dom}(S \cap \alpha I)$. The product of linear relations is associative.

If $S$ is a linear relation in a Pontryagin space $(\mathfrak{F}, [\, \cdot \,, \cdot \,]_{\mathfrak{F}})$, its *adjoint* $S^*$ is defined by

$$S^* = \big\{\{u, v\} \in \mathfrak{F}^2 \ : \ [v, x]_{\mathfrak{F}} - [u, y]_{\mathfrak{F}} = 0 \text{ for all } \{x, y\} \in S\big\}.$$

Thus, for example, $S^*(0) = (\text{dom}\, S)^{[\perp]}$. The use of graph notation is inevitable since we study operators $S$ in a finite-dimensional Pontryagin space $(\mathfrak{F}, [\, \cdot \,, \cdot \,]_{\mathfrak{F}})$ which are not

defined on all of $\mathfrak{F}$. In this case $S^*(0) \neq \{0\}$ and hence $S^*$ is not (the graph of) an operator. Finally we define the direct sum of linear relations $S$ and $T$ in $\mathfrak{F}$: if $S \cap T = \{\{0,0\}\}$ we set

$$S \dotplus T := \{\{u + x, v + y\} : \{u, v\} \in S, \{x, y\} \in T\}$$

and call it the *direct sum* of $S$ and $T$.

By 0 we denote either the zero relation $0 = \{\{0,0\}\}$ or the zero operator

$$0 = \{\{u, 0\} \in \mathfrak{F}^2 : u \in \mathfrak{F}\}.$$

The distinction should be clear from the context. If $S$ is a linear relation in $\mathfrak{F}$, then $\dim S$ is well defined. For example, $\dim S^m = 0$, means $S^m = 0 = \{\{0,0\}\}$, that is $(\operatorname{ran} S^{m-1}) \cap (\operatorname{dom} S) = \{0\}$, while for a nilpotent operator $N$, $N^m = 0$ means that $N^m$ is the zero operator. If $S$ is an operator in $\mathfrak{F}$, then $\dim S = \dim(\operatorname{dom} S)$.

Since we often deal with two Pontryagin spaces, the notation for an inner product includes the space in which it acts as a subscript. The exception is made in the notation of the orthogonal complement. If $\mathfrak{L}$ is a subspace of a Pontryagin space $(\mathfrak{F}, [\cdot, \cdot]_{\mathfrak{F}})$, then $\mathfrak{L}^{[\perp]}$ denotes the orthogonal complement of $\mathfrak{L}$ in $\mathfrak{F}$ with respect to $[\cdot, \cdot]_{\mathfrak{F}}$.

The trivial vector space is excluded from our considerations.

## 3. Vector space preliminaries

The lemma below states that the resolvent of a linear operator $A$ on a finite-dimensional vector space scaled by the characteristic polynomial of $A$ is an operator polynomial whose coefficients are linear combinations of powers of $A$.

Let $\mathfrak{F}$ be a finite-dimensional vector space of dimension $n$ with $n \in \mathbb{N}$. For a linear operator $A : \mathfrak{F} \to \mathfrak{F}$ (defined on the whole space $\mathfrak{F}$) we define the *spectrum of $A$*, denoted by $\sigma(A)$, to be the multiset of the eigenvalues of $A$ in which each eigenvalue $\lambda$ of $A$ is repeated $\dim(\ker(A - \lambda)^n)$ times. We define the *characteristic polynomial of $A$* to be

$$p_A(z) := \prod_{\lambda \in \sigma(A)} (z - \lambda) = \sum_{k=0}^{n} (-1)^k \, e_k \, z^{n-k},$$

where $e_k$ with $k \in \{0, 1, \ldots, n\}$ are the elementary symmetric polynomials in $n$ variables taken from the multiset $\sigma(A)$, see [8, Section 1.1 E.2]. In particular the leading coefficient of $p_A$ is $e_0 = 1$. By $\rho(A)$ we denote the *resolvent set of $A$*, that is, the set of complex numbers which are not eigenvalues of $A$.

**Lemma 3.1.** *Let $\mathfrak{F}$ be a finite-dimensional vector space with $\dim \mathfrak{F} = n$. Let $A : \mathfrak{F} \to \mathfrak{F}$ be a linear operator on $\mathfrak{F}$. Then*

$$-p_A(z)(A-z)^{-1} = \sum_{k=0}^{n-1} C_k z^k \quad \text{for all} \quad z \in \rho(A), \tag{3.1}$$

*where*

$$C_{n-k} = \sum_{j=0}^{k-1}(-1)^j\, e_j\, A^{k-1-j} = A^{k-1} - \cdots + (-1)^{k-1}\, e_{k-1}\, I, \quad k \in \{1,\ldots,n\}. \tag{3.2}$$

**Proof.** The proof is by verification: Multiply (3.1) by $A - z$ and substitute (3.2) in the resulting expression. This will yield an identity on the resolvent set of $A$.   □

The following lemma is taken from [5, Lemma 3.2]. We give a slightly different proof.

**Lemma 3.2.** *Let $\mathfrak{F}$ be a vector space and $n \in \mathbb{N}$. Let $S$ be a nonzero operator in $\mathfrak{F}$ which is defined on a subspace* dom $S$ *of $\mathfrak{F}$. Let $z_1, \ldots, z_n$ be distinct complex numbers which are not eigenvalues of $S$. Then*

$$\bigcap_{k=1}^{n} \operatorname{ran}(S - z_k) = \operatorname{ran}\big((S - z_1) \cdots (S - z_n)\big). \tag{3.3}$$

**Proof.** It suffices to prove (3.3) for $n > 1$. The inclusion $\supseteq$ is clear because the operator factors on the right commute. The inclusion $\subseteq$ is proved by induction on $n$. Let $n = 2$ and let $w = (S - z_1)v_1$ and $w = (S - z_2)v_2$ with $v_1, v_2 \in$ dom $S$. Then

$$S(v_1 - v_2) = z_1 v_1 - z_2 v_2 \in \text{dom } S.$$

Therefore $v_1 - v_2 \in$ dom $S^2 = $ dom$\big((S - z_1)(S - z_2)\big)$. Let us now calculate

$$(S - z_2)(S - z_1)(v_1 - v_2) = (S - z_2)(z_1 - z_2)v_2$$
$$= (z_1 - z_2)w.$$

Hence, $w \in \operatorname{ran}\big((S - z_1)(S - z_2)\big)$.

To prove the inductive step, let $n \in \mathbb{N}$, $n > 1$ and assume that (3.3) is true whenever $z_1, \ldots, z_n$ are distinct complex numbers which are not eigenvalues of $S$. Let $\mu_1, \ldots, \mu_n, \mu_{n+1}$ be distinct complex numbers which are not eigenvalues of $S$. Let $w = (S - \mu_k)v_k$ with $v_k \in$ dom $S$ for all $k \in \{1, \ldots, n + 1\}$. For each $k \in \{1, \ldots, n\}$, by what has been proved for $n = 2$, we deduce that there exist $u_k \in$ dom $S^2$ such that $w = (S - z_{n+1})(S - z_k)u_k$. Hence $(S - z_{n+1})^{-1}w$ is in $\operatorname{ran}(S - z_k)$ for all $k \in \{1, \ldots, n\}$. By the inductive hypothesis there exists $u \in$ dom $S^n$ such that

$$(S - z_{n+1})^{-1}w = (S - z_n) \cdots (S - z_1)u.$$

This proves that $w$ is in the range of the operator $(S - z_{n+1})(S - z_n) \cdots (S - z_1)$.   □

**Corollary 3.3.** *Let $S$ be an operator in a finite-dimensional vector space $\mathfrak{F}$ which is defined on a proper subspace $\mathrm{dom}\, S$ of $\mathfrak{F}$ and which has no eigenvalues. Let $\mathbb{F} \subseteq \mathbb{C}$ be a finite set. Then*

$$\dim\Big(\bigcap\{\mathrm{ran}(S - z) \,:\, z \in \mathbb{F}\}\Big) = \dim S^{\#\mathbb{F}}.$$

**Proof.** Assume $n = \#\mathbb{F}$ and $\mathbb{F} = \{z_1, \ldots, z_n\}$. The claim follows from (3.3) and the fact that the operator on the right-hand side of (3.3) is a linear injection defined on $\mathrm{dom}(S^n)$ which has the same dimension as $S^n$.  $\square$

**Lemma 3.4.** *Let $\mathfrak{F}$ be an $n$-dimensional Pontryagin space and let $S : \mathrm{dom}\, S \to \mathfrak{F}$ be an operator in $\mathfrak{F}$ which has no eigenvalues. Set $d = \mathrm{codim}(\mathrm{dom}\, S)$. Then $d \in \mathbb{N}$ and for all $z \in \mathbb{C}$ the spaces $\ker(S^* - z)$ and $S^* \cap zI$ have dimension $d$. Furthermore, $\mathrm{dom}\, S$, $S$ and for all $z \in \mathbb{C}$ the spaces $\mathrm{ran}(S - z)$ have dimension $n - d$.*

**Proof.** Let $z \in \mathbb{C}$ be arbitrary. The spaces $\mathrm{dom}\, S$ and $\mathrm{ran}(S - z^*)$ have the same dimension $n - d$ since $S - z^*$ is a bijection between these spaces. This is also the dimension of $S$ since there is a trivial linear bijection mapping each $v \in \mathrm{dom}\, S$ to the pair $\{v, Sv\} \in S$. Since the orthogonal complement of $\mathrm{ran}(S - z^*)$ is $\ker(S^* - z)$, we have $\dim \ker(S^* - z) = d$. As the equivalence $\{v, zv\} \in S^* \cap zI$ if and only if $v \in \ker(S^* - z)$ establishes a linear bijection between $S^* \cap zI$ and $\ker(S^* - z)$, we also have $\dim\big(S^* \cap zI\big) = d$.  $\square$

The following lemma is used in the first proof of Theorem 11.1 and it is illustrated in Example 12.2.

**Lemma 3.5.** *Let $(\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}})$ be a finite-dimensional Hilbert space and let $S$ be a symmetric operator in $(\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}})$ which is defined on a proper subspace $\mathrm{dom}\, S$ of $\mathfrak{H}$. Denote by $S^*$ the adjoint of $S$ and by $P$ the orthogonal projection onto $\mathrm{dom}\, S$. Then the linear relation*

$$S \dot{+} PS^*\big|_{(\mathrm{dom}\, S)^{\langle\perp\rangle}} = \Big\{\{x + u, y + Pv\} \,:\, \{x, y\} \in S,\ \{u, v\} \in S^*,\ u \in (\mathrm{dom}\, S)^{\langle\perp\rangle}\Big\}$$

*is a self-adjoint operator extension of $S$ defined on all of $\mathfrak{H}$.*

**Proof.** Since the adjoint of $S^*$ is $S$ and $S$ is an operator, the adjoint $S^*$ is defined on the whole space $\mathfrak{H}$. Moreover, $(\mathrm{dom}\, S)^{\langle\perp\rangle} = S^*(0)$. Set

$$S_1 = PS^*\big|_{(\mathrm{dom}\, S)^{\langle\perp\rangle}} = \big\{\{u, Pv\} \,:\, \{u, v\} \in S^*,\ u \in (\mathrm{dom}\, S)^{\langle\perp\rangle}\big\}.$$

Then $S_1$ is an operator with $\mathrm{dom}\, S_1 = (\mathrm{dom}\, S)^{\langle\perp\rangle}$ and since $S^*$ is an extension of $S_1$, for all $x \in \mathrm{dom}\, S$ and all $u \in (\mathrm{dom}\, S)^{\langle\perp\rangle}$ we have

$$\langle Sx, u \rangle - \langle x, S_1 u \rangle = 0.$$

Let $x, y \in \operatorname{dom} S$ and all $u, v \in (\operatorname{dom} S)^{\langle \perp \rangle}$ be arbitrary and calculate using the fact that $S$ is symmetric, the preceding equality and the fact that the range of $S_1$ is in $\operatorname{dom} S$:

$$\langle Sx + S_1 u, y + v \rangle - \langle x + u, Sy + S_1 v \rangle$$
$$= \langle Sx, y \rangle + \langle S_1 u, y \rangle + \langle Sx, v \rangle + \langle S_1 u, v \rangle$$
$$\qquad - \langle x, Sy \rangle - \langle u, Sy \rangle - \langle x, S_1 v \rangle - \langle u, S_1 v \rangle$$
$$= \big( \langle S_1 u, y \rangle - \langle u, Sy \rangle \big) + \big( \langle Sx, v \rangle - \langle x, S_1 v \rangle \big) + \langle S_1 u, v \rangle - \langle u, S_1 v \rangle$$
$$= 0 + 0 + 0 - 0$$
$$= 0.$$

This shows that $S \dotplus S_1$ is symmetric, and from

$$\operatorname{dom}(S \dotplus S_1) = \operatorname{dom}(S) \langle + \rangle \operatorname{dom}(S_1) = \mathfrak{H}$$

it follows that it is self-adjoint. $\square$

We end this section with a proposition about the main topic of our paper, namely an operator without eigenvalues in a finite-dimensional vector space. We show that the sequence of the dimensions of powers of such an operator has a special property which is the key to our main results.

**Proposition 3.6.** *Let $S$ be an operator in a finite-dimensional space $\mathfrak{F}$ which is defined on a proper subspace $\operatorname{dom} S$ of $\mathfrak{F}$. The operator $S$ has no eigenvalues if and only if there exists a positive integer $m$ such that $S^m = 0$. For the smallest $m$ with this property we have $m \leq \dim \mathfrak{F}$ and*

$$\dim \mathfrak{F} = \dim S^0 > \dim S > \cdots > \dim S^{m-1} > \dim S^m = 0. \tag{3.4}$$

*Furthermore,*

$$\dim S^{k-1} - \dim S^k \geq \dim S^k - \dim S^{k+1} \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.5}$$

**Proof.** Notice that by the definition of the composition of operators $\operatorname{dom} S^k \subseteq \operatorname{dom} S^{k-1}$ for all $k \in \mathbb{N}$. Therefore, if $S^k \neq 0$ for all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $\dim S^l = \dim S^{l-1} > 0$. For such $l$, $\operatorname{dom} S^l$ is a nontrivial invariant subspace of $S$, which implies that $S$ has an eigenvalue. The converse, if $S$ has an eigenvalue, then $S^k \neq 0$ for all $k \in \mathbb{N}$ is straightforward. Hence, $S$ has no eigenvalues if and only if $S^m = 0$ for some $m \in \mathbb{N}$.

Now assume that $S$ has no eigenvalues and $S^m = 0$. From the first paragraph of this proof we have that $\dim S^{k-1} > \dim S^k$ whenever $k \in \mathbb{N}$ is such that $\dim S^{k-1} > 0$. Since there are only finitely many positive integers which are smaller than $\dim S^0 = \dim \mathfrak{F}$, we deduce that $m \leq \dim \mathfrak{F}$ and (3.4) holds. Notice that $m = 1$ if $S = 0$.

To prove (3.5) notice the following equalities. For all $l \in \mathbb{N}$ we have

$$S \operatorname{dom} S^l = (\operatorname{dom} S^{l-1}) \cap (\operatorname{ran} S) \quad \text{and} \quad \dim(S \operatorname{dom} S^l) = \dim S^l. \qquad (3.6)$$

Let $k \in \mathbb{N}$ be arbitrary. By (3.6) we have $S \operatorname{dom} S^k \subseteq \operatorname{dom} S^{k-1}$. Together with $\operatorname{dom} S^k \subseteq \operatorname{dom} S^{k-1}$ this yields

$$\dim(\operatorname{dom} S^{k-1}) \geq \dim(S \operatorname{dom} S^k) + \dim(\operatorname{dom} S^k) - \dim((S \operatorname{dom} S^k) \cap (\operatorname{dom} S^k)). \quad (3.7)$$

Applying the first equality in (3.6) twice in opposite directions we get

$$(S \operatorname{dom} S^k) \cap (\operatorname{dom} S^k) = (\operatorname{ran} S) \cap (\operatorname{dom} S^k) = S \operatorname{dom} S^{k+1}.$$

Therefore by the second equality in (3.6)

$$\dim((S \operatorname{dom} S^k) \cap (\operatorname{dom} S^k)) = \dim(S \operatorname{dom} S^{k+1}) = \dim S^{k+1}.$$

Again, by the second equality in (3.6), the inequality (3.7) becomes

$$\dim S^{k-1} \geq 2 \dim S^k - \dim S^{k+1}. \quad \square$$

**Definition 3.7.** Let $S$ be an operator without eigenvalues in a finite-dimensional vector space $\mathfrak{F}$. The tuple of positive dimensions of powers of $S$ in (3.4) will be denoted by $\delta_S$ and it will be called the *tuple of dimensions of* $S$.

Thus, if $\delta_S = (\delta_1, \ldots, \delta_m)$, then

$$\delta_1 = \dim S^0 > \delta_2 = \dim S > \cdots > \delta_m = \dim S^{m-1} > \dim S^m = 0.$$

As a consequence of Corollary 3.3 we have:

**Corollary 3.8.** *Let $S$ be an operator in a finite-dimensional vector space $\mathfrak{F}$ which is defined on a proper subspace $\operatorname{dom} S$ of $\mathfrak{F}$ and which has no eigenvalues. Let $\mathbb{F} \subseteq \mathbb{C}$ be such that $\#\mathbb{F} \geq \#\delta_S$. Then*

$$\bigcap \{\operatorname{ran}(S - z) : z \in \mathbb{F}\} = \{0\}.$$

## 4. Preliminaries about matrix polynomials

In this section $d, m \in \mathbb{N}$. In addition we assume that $d \leq m$, but we notice that the statements of this section hold true if $d > m$ with analogous proofs. Let $\mathcal{S}(z) \in \mathbb{C}^{d \times m}[z]$ with rank $\mathcal{S}(z) = d$ for some $z \in \mathbb{C}$. For $j \in \{1, \ldots, d\}$ let $\sigma_j$ be the degree of the $j$-th row of $\mathcal{S}(z)$. The rank condition rank $\mathcal{S}(z) = d$ for some $z \in \mathbb{C}$ implies that all $\sigma_j \geq 0$.

Associate with $\mathcal{S}(z)$ the $d$-tuple $\Omega_{\mathcal{S}} = (\sigma'_1, \ldots, \sigma'_d)$ which consists of the ordered row degrees of $\mathcal{S}(z)$. That is, $\{\sigma'_1, \ldots, \sigma'_d\} = \{\sigma_1, \ldots, \sigma_d\}$ as multisets and $\sigma'_1 \geq \cdots \geq \sigma'_d \geq 0$.

We introduce a partial order among $d$-tuples of nonnegative integers. For $\alpha, \beta \in (\mathbb{N}_0)^d$ we set

$$\alpha \preceq \beta \iff \alpha_i \leq \beta_i \quad \text{for all} \quad i \in \{1, \ldots, d\}$$

and $\alpha \prec \beta$ if and only if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

Define $S_\infty$ by

$$S_\infty = \lim_{z \to \infty} \operatorname{diag}(z^{-\sigma_1}, \ldots, z^{-\sigma_d}) \, \mathcal{S}(z).$$

Notice that each row of $\mathcal{S}(z)$ is a row vector polynomial in $\mathbb{C}^{1 \times m}[z]$. The definition of $S_\infty$ implies that the rows of $S_\infty$ are the leading coefficients of these row vector polynomials. Thus, the rows of $S_\infty$ are nonzero. If $\operatorname{rank} S_\infty = d$, then the numbers $\sigma_1, \ldots, \sigma_d$ are called the *Forney indices* of $\mathcal{S}(z)$. This definition will be extended to an equivalence class of polynomials after the proof of Theorem 4.3.

The matrix polynomial $\mathcal{S}(z)$ is said to have the *predictable degree property* if for every

$$u(z) = \begin{bmatrix} u_1(z) & \cdots & u_d(z) \end{bmatrix} \in \mathbb{C}^{1 \times d}[z]$$

we have

$$\deg\bigl(u(z)\mathcal{S}(z)\bigr) = \max\{\sigma_j + \deg u_j(z) \,:\, j \in \{1, \ldots, d\}\}.$$

**Theorem 4.1.** *Let $d, m \in \mathbb{N}$ be such that $d \leq m$. Let $\mathcal{P}(z) \in \mathbb{C}^{d \times m}[z]$ be such that $\operatorname{rank} \mathcal{P}(z) = d$ for some $z \in \mathbb{C}$. The following statements are equivalent:*

(a) $\operatorname{rank} P_\infty = d$.
(b) *$\mathcal{P}(z)$ has the predictable degree property.*
(c) *$\Omega_{\mathcal{P}} \preceq \Omega_{\mathcal{A}\mathcal{P}}$ for every $\mathcal{A}(z) \in \mathbb{C}^{d \times d}[z]$ such that $\det \mathcal{A}(z) \not\equiv 0$.*
(d) *$\Omega_{\mathcal{P}} \preceq \Omega_{\mathcal{U}\mathcal{P}}$ for every unimodular $\mathcal{U}(z) \in \mathbb{C}^{d \times d}[z]$.*

**Proof.** The equivalence (a) $\Leftrightarrow$ (b) is [20, Theorem 6.3-13].

(b)$\Rightarrow$(c). Let $\mathcal{A}(z) \in \mathbb{C}^{d \times d}[z]$ be such that $\det \mathcal{A}(z) \not\equiv 0$ and set $\mathcal{T}(z) = \mathcal{A}(z)\mathcal{P}(z)$. Then $\operatorname{rank} \mathcal{T}(z) = d$ for some $z \in \mathbb{C}$. Let $\tau_1, \ldots, \tau_d$ be the row degrees of $\mathcal{T}(z)$ and let $\pi_1, \ldots, \pi_d$ be the row degrees of $\mathcal{P}(z)$. Without loss of generality we can assume that these row degrees are ordered

$$\tau_1 \geq \cdots \geq \tau_d \quad \text{and} \quad \pi_1 \geq \cdots \geq \pi_d. \tag{4.1}$$

That is we assume

$$\Omega_{\mathscr{T}} = (\tau_1, \ldots, \tau_d) \quad \text{and} \quad \Omega_{\mathscr{P}} = (\pi_1, \ldots, \pi_d).$$

Denote by $a_{kl}(z)$, $k, l \in \{1, \ldots, d\}$, the entry of $\mathscr{A}(z)$ in the $k$-th row and $l$-th column. The $k$-th row of $\mathscr{T}(z)$ equals the $k$-th row of $\mathscr{A}(z)$ times $\mathscr{P}(z)$. By (b) $\mathscr{P}(z)$ has the predictable degree property. Therefore, for all $k \in \{1, \ldots, d\}$ we have

$$\tau_k = \max\{\pi_l + \deg a_{kl}(z) : l \in \{1, \ldots, d\}\}. \tag{4.2}$$

Let $j \in \{1, \ldots, d\}$ be arbitrary. Notice that any scalar $d \times d$ matrix with a rectangular zero block of size $(d - j + 1) \times j$ is singular. Therefore, if for all $k \in \{j, \ldots, d\}$ and all $l \in \{1, \ldots, j\}$ we have $a_{kl}(z) \equiv 0$, then $\det \mathscr{A}(z) \equiv 0$. Since $\det \mathscr{A}(z) \not\equiv 0$, there exist $k_0 \in \{j, \ldots, d\}$ and $l_0 \in \{1, \ldots, j\}$ such that $a_{k_0 l_0}(z) \not\equiv 0$. Consequently, using (4.2),

$$\tau_{k_0} = \max\{\pi_l + \deg a_{k_0 l}(z) : l \in \{1, \ldots, d\}\} \geq \pi_{l_0}.$$

Now, (4.1) yields

$$\tau_j \geq \tau_{k_0} \geq \pi_{l_0} \geq \pi_j.$$

Since $j \in \{1, \ldots, d\}$ was arbitrary, we proved $\Omega_{\mathscr{P}} \preceq \Omega_{\mathscr{T}}$.

(c)$\Rightarrow$(d) is trivial.

We prove (d)$\Rightarrow$(a) by proving its contrapositive. Introduce the following notation. Set

$$\mathscr{P}_\infty(z) := \operatorname{diag}(z^{\pi_1}, \ldots, z^{\pi_d}) P_\infty$$

and

$$\mathscr{R}(z) = \mathscr{P}(z) - \mathscr{P}_\infty(z).$$

Denote by $\rho_1, \ldots, \rho_d$ the degrees of the rows of $\mathscr{R}(z)$. Then, by the definition of $P_\infty$ we have

$$\pi_k > \rho_k \quad \text{for all} \quad k \in \{1, \ldots, d\}.$$

Without loss of generality we assume that the row degrees of $\mathscr{P}(z)$ are ordered:

$$\pi_1 \geq \cdots \geq \pi_d. \tag{4.3}$$

Assume that $\operatorname{rank} P_\infty < d$. Then there exists a nonzero $a \in \mathbb{C}^{1 \times d}$ such that $a P_\infty = 0$ and

$$a = [0 \cdots 0 \; 1 \; a_{i+1} \cdots a_d]$$

for some $i \in \{1, \ldots, d-1\}$; the case $a = [0 \cdots 0 \ 1]$ is not possible since the rows of $P_\infty$ are nonzero.

Let $\mathcal{U}(z)$ be the unimodular matrix which is obtained by replacing the $i$-th row of the identity matrix $I_d$ with the following row vector polynomial

$$\begin{bmatrix} 0 \cdots 0 & 1 & a_{i+1} z^{\pi_i - \pi_{i+1}} \cdots a_d z^{\pi_i - \pi_d} \end{bmatrix}.$$

For $k \in \{1, \ldots, d\}$ let $\theta_k$ be the degree of the $k$-th row of $\mathcal{U}(z)\mathcal{P}(z)$. Since all the rows except the $i$-th row of $\mathcal{U}(z)\mathcal{P}(z)$ are identical with the rows of $\mathcal{P}(z)$, we have

$$\theta_k = \pi_k \quad \text{for all} \quad k \in \{1, \ldots, d\} \setminus \{i\}. \tag{4.4}$$

The $i$-th row of $\mathcal{U}(z)\mathcal{P}_\infty(z)$ is

$$a P_\infty z^{\pi_i} = 0,$$

while the $i$-th row of $\mathcal{U}(z)\mathcal{R}(z)$ is

$$\mathcal{R}(z)\big|_i + \sum_{k=i+1}^{d} a_k z^{\pi_i - \pi_k} \mathcal{R}(z)\big|_k, \tag{4.5}$$

where $\mathcal{R}(z)\big|_k$ denotes the $k$-th row of $\mathcal{R}(z)$. Since for all $k \in \{i, \ldots, d\}$ we have $\rho_k = \deg \mathcal{R}(z)\big|_k < \pi_k$, the degree of the polynomial in (4.5) is $< \pi_i$. Consequently $\theta_i < \pi_i$.

Set $j = \max\{k \in \{1, \ldots, d\} : \theta_i < \pi_k\}$. Clearly $i \le j$. If $j = i$, then (4.3), (4.4), $\theta_i < \pi_i$, yield that the $d$-tuple $(\theta_1, \ldots, \theta_d)$ is ordered and thus $\Omega_{\mathcal{U}\mathcal{P}} \prec \Omega_{\mathcal{P}}$. If $i < j$, then permute the $d$-tuple $(\theta_1, \ldots, \theta_d)$ to $(\theta_1', \ldots, \theta_d')$ as follows:

$$\theta_k' = \begin{cases} \theta_k = \pi_k & \text{if} \quad k \in \{1, \ldots, i-1\} \cup \{j+1, \ldots, d\}, \\ \theta_{k+1} = \pi_{k+1} & \text{if} \quad k \in \{i, \ldots, j-1\}, \\ \theta_i & \text{if} \quad k = j. \end{cases}$$

It follows from the definition of $j$, (4.3) and $\theta_i < \pi_i$ that the $d$-tuple $(\theta_1', \ldots, \theta_d')$ is ordered $\theta_1' \ge \cdots \ge \theta_d'$. Further, (4.3) and $\theta_i < \pi_i$ yield that $\Omega_{\mathcal{U}\mathcal{P}} \preceq \Omega_{\mathcal{P}}$. Since (4.4) and $\theta_i < \pi_i$ imply that

$$\sum_{k=1}^{d} \theta_k' = \sum_{k=1}^{d} \theta_k < \sum_{k=1}^{d} \pi_k,$$

we deduce that $\Omega_{\mathcal{U}\mathcal{P}} \prec \Omega_{\mathcal{P}}$, proving that (d) is false. $\quad\square$

**Corollary 4.2.** *Let $\mathcal{P}(z), \mathcal{T}(z) \in \mathbb{C}^{d \times m}[z]$ be such that $\operatorname{rank} \mathcal{P}(z) = d$ for some $z \in \mathbb{C}$, $\operatorname{rank} P_\infty = d$ and $\mathcal{P}(z) = \mathcal{V}(z)\mathcal{T}(z)$ for all $z \in \mathbb{C}$ and for some unimodular $\mathcal{V}(z) \in \mathbb{C}^{d \times d}[z]$. Then $\Omega_{\mathcal{T}} = \Omega_{\mathcal{P}}$ if and only if $\operatorname{rank} T_\infty = d$.*

**Proof.** Notice that rank $\mathcal{T}(z) = d$ for some $z \in \mathbb{C}$. Assume $\Omega_{\mathcal{T}} = \Omega_{\mathcal{P}}$. Let $\mathcal{U}(z) \in \mathbb{C}^{d \times d}[z]$ be unimodular. Then $\mathcal{U}(z)\mathcal{T}(z) = \mathcal{U}(z)\mathcal{V}(z)^{-1}\mathcal{P}(z)$. Since $\mathcal{U}(z)\mathcal{V}(z)^{-1}$ is unimodular and rank $P_\infty = d$, we have $\Omega_{\mathcal{T}} = \Omega_{\mathcal{P}} \preceq \Omega_{\mathcal{U}\mathcal{V}^{-1}\mathcal{P}} = \Omega_{\mathcal{U}\mathcal{T}}$. Hence, rank $T_\infty = d$ proving the "only if" part.

Assume rank $T_\infty = d$. Then $\Omega_{\mathcal{T}} \preceq \Omega_{\mathcal{V}\mathcal{T}} = \Omega_{\mathcal{P}}$ and $\Omega_{\mathcal{P}} \preceq \Omega_{\mathcal{V}^{-1}\mathcal{P}} = \Omega_{\mathcal{T}}$, yielding $\Omega_{\mathcal{T}} = \Omega_{\mathcal{P}}$. $\quad\square$

In the proof of the following theorem we use that any nonzero $\mathcal{P}(z) \in \mathbb{C}^{d \times m}[z]$ admits a *Smith normal form* representation (see for example [15, Satz 6.3] or [20, Theorem 6.3-16]):

$$\mathcal{P}(z) = \mathcal{U}(z) \begin{bmatrix} \mathcal{D}(z) & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}(z), \qquad (4.6)$$

where $\mathcal{U}(z) \in \mathbb{C}^{d \times d}[z]$, $\mathcal{V}(z) \in \mathbb{C}^{m \times m}[z]$ are unimodular and the matrix in the middle is a $d \times m$ matrix in which, for some $l \in \{1, \ldots, d\}$, $\mathcal{D}(z)$ is a diagonal $l \times l$ matrix polynomial with monic diagonal entries: $\mathcal{D}(z) = \mathrm{diag}\big(p_1(z), \ldots, p_l(z)\big)$ such that $p_i(z)$ divides $p_{i+1}(z)$ for all $i \in \{1, \ldots, l-1\}$. Notice that rank $\mathcal{P}(\alpha) = l$ if and only if $p_l(\alpha) \neq 0$. If for some $z \in \mathbb{C}$ the rank of $\mathcal{P}(z)$ is $d$, then $l = d$ and the zero block row in the matrix in the middle of the right-hand side of the equality (4.6) is not present.

The matrix in the middle of the right-hand side of (4.6) is uniquely determined by $\mathcal{P}(z)$.

**Theorem 4.3.** *Let $\mathcal{T}(z) \in \mathbb{C}^{d \times m}[z]$ be such that* rank $\mathcal{T}(z) = d$ *for some $z \in \mathbb{C}$. Then $\mathcal{T}(z)$ admits the factorization:*

$$\mathcal{T}(z) = \mathcal{W}(z)\mathcal{P}(z) \quad \text{for all} \quad z \in \mathbb{C},$$

*where $\mathcal{W}(z) \in \mathbb{C}^{d \times d}[z]$, $\det \mathcal{W}(z) \not\equiv 0$ and $\mathcal{P}(z) \in \mathbb{C}^{d \times m}[z]$ is such that*

$$\text{rank } \mathcal{P}(z) = d \quad \text{for all} \quad z \in \mathbb{C}, \quad \text{rank } P_\infty = d \quad \text{and} \quad \pi_1 \geq \cdots \geq \pi_d, \qquad (4.7)$$

*$\pi_1, \ldots, \pi_d$ being the row degrees of $\mathcal{P}(z)$. This factorization is essentially unique, meaning that if also $\mathcal{T}(z) = \mathcal{W}_1(z)\mathcal{P}_1(z)$ for all $z \in \mathbb{C}$, where $\mathcal{W}_1(z)$ and $\mathcal{P}_1(z)$ have the same properties as $\mathcal{W}(z)$ and $\mathcal{P}(z)$, then for some unimodular $d \times d$ matrix polynomial $\mathcal{V}(z)$ we have $\mathcal{W}_1(z) = \mathcal{W}(z)\mathcal{V}(z)^{-1}$ and $\mathcal{P}_1(z) = \mathcal{V}(z)\mathcal{P}(z)$ for all $z \in \mathbb{C}$. In particular, $\Omega_{\mathcal{P}_1} = \Omega_{\mathcal{P}}$.*

**Proof.** Denote by $\mathfrak{R}$ the set of all matrix polynomials $\mathcal{R}(z) \in \mathbb{C}^{d \times m}[z]$ such that rank $\mathcal{R}(z) = d$ for all $z \in \mathbb{C}$ and for which there exists $\mathcal{A}(z) \in \mathbb{C}^{d \times d}[z]$ with $\det \mathcal{A}(z) \not\equiv 0$ such that $\mathcal{T}(z) = \mathcal{A}(z)\mathcal{R}(z)$ for all $z \in \mathbb{C}$.

From the Smith normal form of $\mathcal{T}(z)$ (see [10, Lemma 2.4]) it follows that $\mathfrak{R} \neq \emptyset$.

Define the function $\omega : \mathfrak{R} \to \mathbb{N}$ by

$$\omega(\mathscr{R}) = \sum \Omega_{\mathscr{R}} \quad \text{for all} \quad \mathscr{R}(z) \in \mathfrak{R}.$$

It follows from the Well Ordering Axiom of $\mathbb{Z}$ that $\omega$ takes the minimum value on $\mathfrak{R}$. Let $\mathscr{P}(z) \in \mathfrak{R}$ be such that $\omega(\mathscr{P}) \le \omega(\mathscr{R})$ for all $\mathscr{R}(z) \in \mathfrak{R}$. This $\mathscr{P}(z)$ has the first two properties asserted in (4.7). The first property follows from $\mathscr{P}(z) \in \mathfrak{R}$. In the proof of (d)⇒(a) in Theorem 4.1 we proved that if rank $P_\infty < d$, then there exists a unimodular $\mathcal{U}(z) \in \mathbb{C}^{d \times d}[z]$ such that $\Omega_{\mathcal{U}\mathscr{P}} \prec \Omega_{\mathscr{P}}$ and consequently $\omega(\mathcal{U}\mathscr{P}) < \omega(\mathscr{P})$. Since $\mathcal{U}\mathscr{P} \in \mathfrak{R}$, this shows that rank $P_\infty < d$ and $\omega(\mathscr{P}) \le \omega(\mathscr{R})$ for all $\mathscr{R}(z) \in \mathfrak{R}$ is not possible. Hence rank $P_\infty = d$. The last property in (4.7) is easily obtained by multiplying $\mathscr{P}(z)$ by a $d \times d$ permutation matrix.

The essential uniqueness follows from the Smith normal form of $\mathscr{P}_1(z)$ and $\mathscr{P}(z)$. The last claim follows from Corollary 4.2.  □

The numbers $\pi_1, \ldots, \pi_d$ in Theorem 4.3 which are the Forney indices of $\mathscr{P}(z)$ are called the *Forney indices* of the matrix polynomial $\mathcal{T}(z)$, see [23] and [13].

## 5. Finite nonincreasing sequences of positive integers

We start with an example. Consider three tuples of positive integers:

$$\mu = (5, 4, 4, 1), \quad \nu = (4, 3, 3, 3, 1), \quad \delta = (14, 10, 7, 4, 1).$$

Each of these tuples is represented on the *Young diagram* (which is also called Ferrers board), see [29, page 58] or [3, Section 3.1], in Fig. 1: the tuple $\mu$ is represented by the number of squares in the rows starting from the top row; the tuple $\nu$ is represented by the number of squares in the columns starting from the leftmost column, and the tuple $\delta$ is represented by the number of squares of different shades of gray.

Each of the three given tuples determines the corresponding Young diagram uniquely. This is clear for $\mu$ and $\nu$ and it becomes clear for $\delta$ when we observe that the numbers in $\nu$ are the differences of the consecutive values in $\delta$ (temporarily extended with a ghost zero at the end). To make the relationship between the three given tuples more formal we introduce three operators $\mathsf{Con}$, $\mathsf{Int}$ and $\mathsf{Der}$ on finite nonincreasing tuples of positive integers. With these three operators we have

$$\nu = \mathsf{Con}\,\mu = \mathsf{Der}\,\delta, \qquad \delta = \mathsf{Int}\,\nu = \mathsf{Int}(\mathsf{Con}\,\mu), \qquad \mu = \mathsf{Con}\,\nu = \mathsf{Con}(\mathsf{Der}\,\delta).$$

Before defining these operators let us point out some immediate relationships between these tuples.

$$\max \delta = \sum \mu = \sum \nu, \quad \max \nu = \#\mu, \quad \max \mu = \#\delta = \#\nu.$$
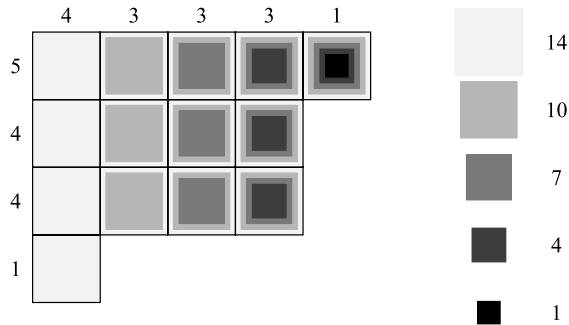
**Fig. 1.** Young diagram.

Let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be a nonincreasing tuple of positive integers. We allow $m = 1$ and still call this 1-tuple nonincreasing. Here $\#\sigma = m$ and we set $n = \max \sigma = \sigma_1$. The *conjugate tuple* $\mathsf{Con}\,\sigma$ is defined as follows:

$$(\mathsf{Con}\,\sigma)_k := \#\{i \in \{1, \ldots, m\} : \sigma_i \geq k\}, \quad k \in \{1, \ldots, n\}.$$

From this definition it immediately follows that

$$\#(\mathsf{Con}\,\sigma) = \max \sigma \quad \text{and} \quad \max(\mathsf{Con}\,\sigma) = \#\sigma.$$

The *integral operator* $\mathsf{Int}$ is defined as

$$(\mathsf{Int}\,\sigma)_j = \sum_{i=j}^{m} \sigma_i, \quad j \in \{1, \ldots, m\}.$$

Clearly,

$$\#(\mathsf{Int}\,\sigma) = \#\sigma \quad \text{and} \quad \max(\mathsf{Int}\,\sigma) = \sum \sigma.$$

Notice that since $\sigma$ is a nonincreasing tuple of positive integers, the tuple $\tau = \mathsf{Int}\,\sigma$ is decreasing and $\tau$ has the property

$$\tau_{j-1} - \tau_j \geq \tau_j - \tau_{j+1}, \quad j \in \{2, \ldots, m\}, \tag{5.1}$$

where we temporarily assign $\tau_{m+1} = 0$ for convenience. A tuple which satisfies (5.1) is said to be *concave up*. By definition we say that a 1-tuple is concave up.

For a concave up tuple $\tau$ we define the *derivative operator* $\mathsf{Der}$:

$$(\mathsf{Der}\,\tau)_j = \tau_j - \tau_{j+1}, \quad j \in \{1, \ldots, m\} \quad \text{where} \quad \tau_{m+1} = 0.$$

The assumption (5.1) guarantees that the tuple $\mathsf{Der}\,\tau$ is nonincreasing. We clearly have $\mathsf{Der}(\mathsf{Int}\,\sigma) = \sigma$ for all nonincreasing tuples of positive integers $\sigma$.

For completeness we prove that Con is an involution on the set of all nonincreasing tuples of positive integers.

**Proposition 5.1.** *For all nonincreasing tuples $\sigma$ of positive integers we have*

$$\mathsf{Con}(\mathsf{Con}\,\sigma) = \sigma.$$

**Proof.** Set $m = \#\sigma$ and $n = \max\sigma$. Since $\sigma$ is nonincreasing, the following two implications are clear: For all $j \in \{1,\ldots,m\}$ and all $l \in \{1,\ldots,n\}$ we have

$$\sigma_j \geq l \Rightarrow (\mathsf{Con}\,\sigma)_l \geq j \qquad \text{and} \qquad \sigma_j < l \Rightarrow (\mathsf{Con}\,\sigma)_l < j.$$

Set $l = \sigma_j$. Then by the first implication $(\mathsf{Con}\,\sigma)_l \geq j$. Applying the same implication to the tuple $\mathsf{Con}\,\sigma$ yields

$$\big(\mathsf{Con}(\mathsf{Con}\,\sigma)\big)_j \geq l.$$

Since $\sigma_j < l + 1$, applying the second implication gives $(\mathsf{Con}\,\sigma)_{l+1} < j$, and repeating this implication for the tuple $\mathsf{Con}\,\sigma$ yields

$$\big(\mathsf{Con}(\mathsf{Con}\,\sigma)\big)_j < l + 1.$$

The last two displayed inequalities imply the asserted equality.   □

In number theory Young diagrams are commonly used to visualize the number of distinct ways of representing a positive integer as a sum of positive integers; so called *integer partitions*. For a positive integer $n$ the number of its partitions is denoted by $\mathsf{p}(n)$. No elementary explicit formula exists for $\mathsf{p}(n)$ in terms of $n$. Euler found the generating function for $\mathsf{p}(n)$:

$$\sum_{n=0}^{\infty} \mathsf{p}(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots,$$

which is justified by expending each factor in the product in a geometric series, see [3]. Hardy and Ramanujan discovered the asymptotic relation

$$\mathsf{p}(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{2n/3}\right) \quad \text{as} \quad n \to +\infty,$$

see [4,21,24,25], for more details.

The number 5 can be represented in $\mathsf{p}(5) = 7$ ways:



| 1+1+1+1+1 | 2+1+1+1 | 3+1+1 | 2+2+1 | 4+1 | 3+2 | 5 |
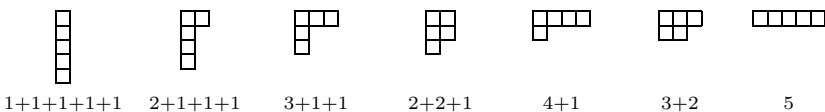
Here we have counted boxes in each row to get the summands adding to 5. However, we could also have counted boxes in each column and get the same representations in different order. In number theory partitions that are connected as $\nu = \mathsf{Con}\,\mu$ are called *conjugate partitions*. That is the source for this operator's name.

In the next section we will see that Young diagrams can be used to represent canonical subspaces of $\mathbb{C}^d[z]$. In this setting: (a) the number of boxes in each row of a Young diagram is equal to the highest degree of a polynomial in that row plus one, (b) the number of boxes in the $k$-th column is equal to the maximum number of nonzero entries in a vector coefficient with the power $z^{k-1}$, (c) the number of boxes that we shaded in the same shade of gray in Fig. 1 is equal to the dimension of the range of a specific power of multiplication by the independent variable (notice that since this operator is bijective the dimensions of the ranges are equal to the dimensions of the domains of the same powers).

## 6. Canonical subspaces of $\mathbb{C}^d[z]$ and Young diagrams

**Definition 6.1.** Let $d \in \mathbb{N}$ and $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ with $\mu_1 \geq \cdots \geq \mu_d$. Set

$$
\begin{aligned}
\mathfrak{C}_\mu &:= \bigoplus_{k=1}^{d} \left( \mathbb{C}[z]_{<\mu_k} \right) e_{d,k} \\
&= \left\{ [p_1(z) \cdots p_d(z)]^\top : p_k(z) \in \mathbb{C}[z],\ \deg p_k < \mu_k,\ k \in \{1, \ldots, d\} \right\}.
\end{aligned}
$$

The tuple $\mu = (\mu_1, \ldots, \mu_d)$ will be called the *tuple of degrees of* $\mathfrak{C}_\mu$. The space $\mathfrak{C}_\mu$ is called a *canonical subspace of* $\mathbb{C}^d[z]$ where $d = \#\mu$, or simply a *canonical space* of vector polynomials. The set of vector polynomials

$$
\left\{ z^l e_{d,k} : l \in \{0, 1, \ldots, \mu_k - 1\},\ k \in \{1, \ldots, d\} \right\}
$$

is called the *standard basis* for $\mathfrak{C}_\mu$.

Notice here that the maximal degrees of the polynomial entries of vector polynomials in $\mathfrak{C}_\mu$ are $\mu_1 - 1, \ldots, \mu_d - 1$. If $\mathfrak{C}$ stands for a canonical space, then its degrees are either clear from the context or not important. We set

$$
d = \#\mu, \quad m = \mu_1 = \max \mu \qquad \text{and} \qquad n = \dim \mathfrak{C}_\mu = \sum \mu = \mu_1 + \cdots + \mu_d.
$$

The canonical subspace $\mathfrak{C}_\mu$ is represented by a Young diagram with $\mu_j$ boxes in the $j$-th row. The conjugate tuple $\nu = \mathsf{Con}\,\mu = (\nu_1, \ldots, \nu_m)$ relates to polynomials in $\mathfrak{C}_\mu$ in the following way. Let $f(z) \in \mathfrak{C}_\mu \subset \mathbb{C}^d[z]$ be written as
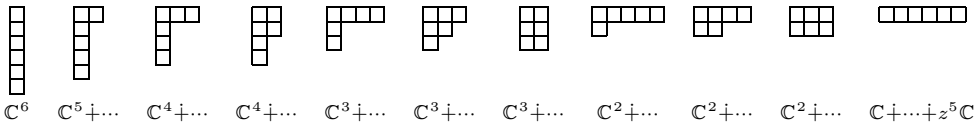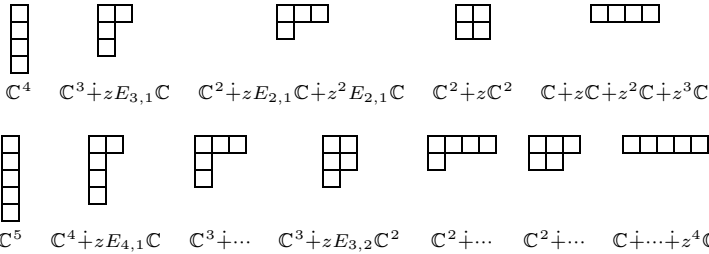
$$
f(z) = f_0 + f_1 z + \cdots + f_{m-1} z^{m-1} \quad \text{with} \quad f_{k-1} \in \mathbb{C}^d \quad \text{for} \quad k \in \{1, \ldots, m\}.
$$

Then all the entries of the column vector $f_{k-1}$ with indices $> \nu_k$ are zero. In particular, the vector $f_0$ can be any vector in $\mathbb{C}^d$ since $\nu_1 = d$. Consequently,

$$\mathfrak{C}_\mu = \left\{ \sum_{j=1}^m z^{j-1} E_{d,\nu_j} a_j \ : \ a_j \in \mathbb{C}^{\nu_j}, \ j \in \{1, \ldots, d\} \right\}.$$

For each $n \in \mathbb{N}$ there is a bijection between the integer partitions of $n$ and the canonical spaces of dimension $n$. Therefore there are exactly $\mathsf{p}(n)$ canonical spaces of dimension $n$. For these $\mathsf{p}(n)$ canonical spaces the ambient spaces are $\mathbb{C}^d[z]$ with $d \in \{1, \ldots, n\}$.

Below we show the Young diagrams of all canonical spaces of dimensions 4, 5 and 6. The ambient spaces here are $\mathbb{C}^d[z]$ with $d \in \{1, \ldots, 6\}$. Within each dimension diagrams are ordered by decreasing $d$.



$\mathbb{C}^4$    $\mathbb{C}^3 \dot{+} z E_{3,1} \mathbb{C}$    $\mathbb{C}^2 \dot{+} z E_{2,1} \mathbb{C} \dot{+} z^2 E_{2,1} \mathbb{C}$    $\mathbb{C}^2 \dot{+} z \mathbb{C}^2$    $\mathbb{C} \dot{+} z \mathbb{C} \dot{+} z^2 \mathbb{C} \dot{+} z^3 \mathbb{C}$

$\mathbb{C}^5$    $\mathbb{C}^4 \dot{+} z E_{4,1} \mathbb{C}$    $\mathbb{C}^3 \dot{+} \cdots$    $\mathbb{C}^3 \dot{+} z E_{3,2} \mathbb{C}^2$    $\mathbb{C}^2 \dot{+} \cdots$    $\mathbb{C}^2 \dot{+} \cdots$    $\mathbb{C} \dot{+} \cdots \dot{+} z^4 \mathbb{C}$

$\mathbb{C}^6$   $\mathbb{C}^5 \dot{+} \cdots$   $\mathbb{C}^4 \dot{+} \cdots$   $\mathbb{C}^4 \dot{+} \cdots$   $\mathbb{C}^3 \dot{+} \cdots$   $\mathbb{C}^3 \dot{+} \cdots$   $\mathbb{C}^3 \dot{+} \cdots$   $\mathbb{C}^2 \dot{+} \cdots$   $\mathbb{C}^2 \dot{+} \cdots$   $\mathbb{C}^2 \dot{+} \cdots$   $\mathbb{C} \dot{+} \cdots \dot{+} z^5 \mathbb{C}$

Notice that the maximal degree of polynomials in each row is obtained when 1 is subtracted from the number of boxes in that row: one box constants, two boxes affine functions, three boxes at most quadratic polynomials, et cetera.

As we pointed out in Section 5, a tuple $\mu$ of nonincreasing row degrees or, equivalently, the corresponding conjugate tuple $\nu = \mathsf{Con}(\mu)$ of power depths, of a canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$ uniquely determines the tuple $\delta = \mathsf{Int}\,\nu = \mathsf{Int}(\mathsf{Con}\,\mu)$. In this setting the tuple $\delta$ consists of the nonzero dimensions of the powers of the operator $S_{\mathfrak{C}_\mu}$:

$$\delta_k = \dim(S_{\mathfrak{C}})^{k-1} = \sum_{j=k}^d \nu_j = \sum_{j=1}^d \max\{\mu_j - k, 0\} \quad \text{for} \quad k \in \{1, \ldots, \max \mu\}.$$

The tuple $\delta$ of the positive dimensions of the powers of $S_{\mathfrak{C}}$ is concave up since it is given as $\delta = \mathsf{Int}(\mathsf{Con}\,\mu)$. This is just a confirmation of what has been proven in

Proposition 3.6: The tuple of the positive dimensions of powers of an arbitrary operator without eigenvalues in a finite-dimensional vector space is concave up.

It is important to note that the tuple $\delta$ of the positive dimensions of the powers of $S_{\mathfrak{C}}$ uniquely determines the canonical space $\mathfrak{C} = \mathfrak{C}_\mu$ where $\mu = \mathsf{Con}(\mathsf{Der}\,\delta)$.

We call two canonical subspaces *equivalent* if there exists a linear bijection between them. This equivalence relation splits the family of canonical spaces in equivalence classes of canonical spaces of the same dimension. Earlier we pointed out that for $n \in \mathbb{N}$ the equivalence class of the canonical spaces of dimension $n$ has $\mathsf{p}(n)$ elements. Refine this equivalence relation by requesting that the linear bijection intertwines the corresponding operators of multiplication by the independent variable. The next theorem shows that the corresponding equivalence classes are singletons.

**Theorem 6.2.** *Let $\mathfrak{C}$ and $\mathfrak{D}$ be canonical spaces of vector polynomials and let $S_{\mathfrak{C}}$ and $S_{\mathfrak{D}}$ be the corresponding operators of multiplication by the independent variable in these spaces. Let $\Phi : \mathfrak{C} \to \mathfrak{D}$ be a linear mapping. The following statements are equivalent.*

(I) *$\Phi$ is a linear bijection such that $\Phi S_{\mathfrak{C}} = S_{\mathfrak{D}}\Phi$.*
(II) *There exist*

  (a) *$d \in \mathbb{N}$ and $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ with $\mu_1 \geq \cdots \geq \mu_d$ such that $\mathfrak{C} = \mathfrak{D} = \mathfrak{C}_\mu \subset \mathbb{C}^d[z]$,*
  (b) *a $d \times d$ unimodular matrix polynomial $\mathcal{W}(z) = \big[w_{jk}(z)\big]_{j,k=1}^d$ with*

$$\deg w_{jk}(z) \leq \mu_j - \mu_k \quad \text{for all} \quad j, k \in \{1, \ldots, d\}$$

  *such that $(\Phi f)(z) = \mathcal{W}(z)f(z)$ for all $f \in \mathfrak{C}$.*

**Proof.** Assume (I). The assumption $\Phi S_{\mathfrak{C}} = S_{\mathfrak{D}}\Phi$ implies that $\Phi(S_{\mathfrak{C}})^k = (S_{\mathfrak{D}})^k\Phi$ for all $k \in \mathbb{N}_0$. Since $\Phi$ is a bijection, we have $\dim(S_{\mathfrak{C}})^k = \dim(S_{\mathfrak{D}})^k$ for all $k \in \mathbb{N}_0$. Let $\delta$ be the tuple of positive dimensions of the operators $S_{\mathfrak{C}}$ and $S_{\mathfrak{D}}$. As the degrees of a canonical space are uniquely determined by the dimensions of the powers of the operator of multiplication, it follows that $\mathfrak{C} = \mathfrak{D} = \mathfrak{C}_\mu$, where $\mu = \mathsf{Con}(\mathsf{Der}\,\delta)$. This proves (a).

Next we prove (b). Since the set of vector polynomials

$$\big\{e_{d,k}z^l \ : \ l \in \{0, 1, \ldots, \mu_k - 1\}, \ k \in \{1, \ldots, d\}\big\}$$

is the standard basis for $\mathfrak{C} = \mathfrak{C}_\mu$ and since $\Phi(e_{d,k}z^l) = z^l\Phi(e_{d,k})$ for all $l \in \{1, \ldots, \mu_k - 1\}$, to understand the action of $\Phi$ on $\mathfrak{C}$ it is sufficient to understand $\Phi(e_{d,k})$ for $k \in \{1, \ldots, d\}$.

Let $d' \in \{1, \ldots, d\}$ be the cardinality of the set $\{\mu_1, \ldots, \mu_d\}$. Assume the set equality $\{\mu_1, \ldots, \mu_d\} = \{\mu'_1, \ldots, \mu'_{d'}\}$ and let $i_k \in \{1, \ldots, d\}$ be the multiplicity of $\mu'_k$ in the multiset $\{\mu_1, \ldots, \mu_d\}$, with $k \in \{1, \ldots, d'\}$. We keep the order $\mu'_1 > \cdots > \mu'_{d'}$ and notice that $\sum_{k=1}^{d'} i_k = d$.

In the rest of this proof we write $x \in \mathbb{C}^d$ in the block form

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{d'} \end{bmatrix} \quad \text{where} \quad x_k \in \mathbb{C}^{i_k} \quad \text{for} \quad k \in \{1, \ldots, d'\}.$$

When convenient we will write the above block vector $x$ in a partially transposed form as $[x_1 \ \cdots \ x_{d'}]^\top \in \mathbb{C}^d$. We will use a similar convention for the vector polynomials in $\mathfrak{C}$. With this convention $\mathfrak{C}$ can be described as the set of all vector polynomials $[p_1(z) \ \cdots \ p_{d'}(z)]^\top \in \mathbb{C}^d[z]$ such that $p_k(z) \in \mathbb{C}^{i_k}[z]_{<\mu'_k}$ for all $k \in \{1, \ldots, d'\}$.

To reveal the block structure of $\Phi$ we will use the fact that

$$\Phi\big(\operatorname{dom} S_{\mathfrak{C}}^k + \operatorname{ran} S_{\mathfrak{C}}^l\big) = \operatorname{dom} S_{\mathfrak{C}}^k + \operatorname{ran} S_{\mathfrak{C}}^l \quad \text{for all} \quad k, l \in \mathbb{N}_0.$$

Notice that, since $\mu'_1 > \cdots > \mu'_{d'}$, we have $S_{\mathfrak{C}}^{\mu'_1 - 1} \neq 0$ and $S_{\mathfrak{C}}^{\mu'_1} = 0$. Also, $[p_1(z) \ \cdots \ p_{d'}(z)]^\top \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_1 - 1}$ if and only if there exists $x_1 \in \mathbb{C}^{i_1}$ such that

$$[p_1(z) \ \cdots \ p_{d'}(z)]^\top = [x_1 \ 0 \ \cdots \ 0]^\top.$$

Since $\Phi$ is a linear bijection on $\operatorname{dom} S_{\mathfrak{C}}^{\mu'_1 - 1}$, there exists an $i_1 \times i_1$ invertible matrix $W_{11}$ such that $\Phi[x_1 \ 0 \ \cdots \ 0]^\top = [W_{11}x_1 \ 0 \ \cdots \ 0]^\top$.

Let $x_2 \in \mathbb{C}^{i_2}$ be an arbitrary nonzero vector. Set $x = [0 \ x_2 \ 0 \ \cdots \ 0]^\top$. Then $x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2 - 1}$ and $x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2} + \operatorname{ran} S_{\mathfrak{C}}$. Therefore $\Phi x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2 - 1}$ and $\Phi x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2} + \operatorname{ran} S_{\mathfrak{C}}$. Since $\Phi x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2 - 1}$, there exists $y_2 \in \mathbb{C}^{i_2}$ and $p_1(z) \in \mathbb{C}^{i_1}[z]_{\leq \mu'_1 - \mu'_2}$ such that $\Phi x = [p_1(z) \ y_2 \ 0 \ \cdots \ 0]^\top$. As $\Phi x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_2} + \operatorname{ran} S_{\mathfrak{C}}$ it is not possible that $y_2 = 0$. Consequently, there exists an $i_2 \times i_2$ invertible matrix $W_{22}$ and an $i_1 \times i_2$ matrix polynomial $\mathscr{W}_{12}(z)$ of degree not exceeding $\mu'_1 - \mu'_2$ such that $\Phi x = [\mathscr{W}_{12}(z)x_2 \ W_{22}x_2 \ 0 \ \cdots \ 0]^\top$.

In general, let $k \in \{2, \ldots, d'\}$ and let $x_k \in \mathbb{C}^{i_k}$ be an arbitrary nonzero vector. Set $x = [0 \cdots 0 \ x_k \ 0 \cdots 0]^\top$ so that $x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k - 1}$ and $x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k} + \operatorname{ran} S_{\mathfrak{C}}$. Then $\Phi x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k - 1}$ and $\Phi x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k} + \operatorname{ran} S_{\mathfrak{C}}$. Since $\Phi x \in \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k - 1}$, there exists $y_k \in \mathbb{C}^{i_k}$ and vector polynomials $p_j(z) \in \mathbb{C}^{i_j}[z]_{\leq \mu'_j - \mu'_k}$ with $j \in \{1, \ldots, k-1\}$ such that $\Phi x = [p_1(z) \cdots p_{k-1}(z) \ y_k \ 0 \cdots 0]^\top$. As $\Phi x \notin \operatorname{dom} S_{\mathfrak{C}}^{\mu'_k} + \operatorname{ran} S_{\mathfrak{C}}$ it is not possible that $y_k = 0$. Since $\Phi$ is linear, there exists an $i_k \times i_k$ invertible matrix $W_{kk}$ and an $i_j \times i_k$ matrix polynomials $\mathscr{W}_{jk}(z)$ of degree not exceeding $\mu'_j - \mu'_k$ with $j \in \{1, \ldots, k-1\}$ such that $\Phi x = [\mathscr{W}_{1k}(z)x_k \ \cdots \ \mathscr{W}_{k-1,k}(z)x_k \ W_{kk}x_k \ 0 \ \cdots \ 0]^\top$.

This proves that $\Phi f(z) = \mathscr{W}(z)f(z)$ for all $f(z) \in \mathfrak{C}$, where $\mathscr{W}(z)$ is a $d \times d$ matrix polynomial

$$\mathscr{W}(z) = \begin{bmatrix} W_{11} & \mathscr{W}_{12}(z) & \cdots & \mathscr{W}_{1d'}(z) \\ 0 & W_{22} & \cdots & \mathscr{W}_{2d'}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{d'd'} \end{bmatrix} \tag{6.1}$$

which consists of the blocks $\mathcal{W}_{jk}(z)$, $k, l \in \{1, \ldots, d'\}$, such that $\mathcal{W}_{jk}(z) = 0_{i_j \times i_k}$ if $j > k$, $\mathcal{W}_{kk}(z) = W_{kk}$ is an invertible $i_k \times i_k$ constant matrix, and if $j < k$, $\mathcal{W}_{jk}(z)$ is an $i_j \times i_k$ matrix polynomial of degree not exceeding $\mu'_j - \mu'_k$.

The converse is straightforward.  $\square$

## 7. Similarity to an operator of multiplication

The fact that the tuple of the positive dimensions of powers of an arbitrary operator $S$ without eigenvalues in a finite-dimensional vector space is concave up makes it possible to link $S$ to an operator of multiplication by the independent variable in a canonical space of vector polynomials. This is shown in the second part of the next theorem.

**Theorem 7.1.** *Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator in $\mathfrak{F}$ without eigenvalues. Let $\delta_S$ be the tuple of dimensions of $S$. Let $\mu = (\mu_1, \ldots, \mu_d) = \mathsf{Con}(\mathsf{Der}\,\delta_S)$. The following statements hold.*

(I) *There exist vectors $v_1, \ldots, v_d$ in $\mathfrak{F}$ with $v_j \in \mathrm{dom}\, S^{\mu_j - 1}$, $j \in \{1, \ldots, d\}$, such that the set*

$$\left\{ v_j, \ldots, S^{\mu_j - 1} v_j \ : \ j \in \{1, \ldots, d\} \right\} \tag{7.1}$$

*is a basis for $\mathfrak{F}$.*

(II) *The function defined by*

$$\Phi S^{j-1} v_k = z^{j-1} e_{d,k} \quad \textit{for all} \quad j \in \{1, \ldots, \mu_k\} \quad \textit{and all} \quad k \in \{1, \ldots, d\}$$

*extends linearly to a linear bijection from $\mathfrak{F}$ to $\mathfrak{C}_\mu$ such that $\Phi S = S_{\mathfrak{C}_\mu} \Phi$.*

Recall that $\delta_S = (\delta_1, \ldots, \delta_m)$ is the tuple of positive dimensions $\delta_k = \dim S^{k-1}$ for $k \in \{1, \ldots, m\}$ and $S^m = 0$, where $0$ stands for the zero relation $\{\{0, 0\}\}$. Thus, $\delta_1 = \dim \mathfrak{F}$, $\delta_2 = \dim S$ and $d = \delta_1 - \delta_2 = \mathrm{codim}(\mathrm{dom}\, S)$. Notice that in (7.1) the vectors

$$\left\{ v_j, \ldots, S^{\mu_j - 1} v_j \ : \ j \in \{1, \ldots, d\} \right\} \setminus \left\{ S^{\mu_j - 1} v_j \ : \ j \in \{1, \ldots, d\} \right\} \subset \mathrm{dom}\, S$$

form a basis for $\mathrm{dom}\, S$. There are exactly $-d + \sum_{j=1}^d \mu_j = \delta_2$ vectors in the last set.

**Proof of Theorem 7.1. 1.** If $S = 0$, then $\delta_S = (n)$ where $n = \dim \mathfrak{F}$; that is $\delta_S$ is a tuple consisting of only one positive number. In this case $d = n$ and $\mu = \mathsf{Con}(\mathsf{Der}\,\delta_S) = (1, \ldots, 1)$ is the $n$-tuple consisting of 1s. The theorem is trivially true in this case.

**2.** We assume that $S \neq 0$ and proceed with a proof of (I) by mathematical induction with respect to the cardinality $m = \# \delta_S$. Since $S \neq 0$, we start the induction with $m = 2$.

Assume $S^2 = 0$. Then

$$\delta_1 = \dim S^0 = \dim \mathfrak{F} > \delta_2 = \dim S > \dim S^2 = 0$$

and $\delta_1 - \delta_2 \geq \delta_2$, see (3.5). Notice that

$$d = \delta_1 - \delta_2 = \dim \mathfrak{F} - \dim S = \operatorname{codim}(\operatorname{dom} S).$$

Since $\delta_2 = \dim S = \dim(\operatorname{ran} S)$, there exist $v_1, \ldots, v_{\delta_2} \in \operatorname{dom} S$ such that $Sv_1, \ldots, Sv_{\delta_2}$ form a basis for $\operatorname{ran} S$. Then the $2\delta_2$ vectors

$$v_1, \ldots, v_{\delta_2}, Sv_1, \ldots, Sv_{\delta_2} \tag{7.2}$$

are linearly independent. For a basis for $\mathfrak{F}$ we need additional $\delta_1 - 2\delta_2 = d - \delta_2$ vectors. Let $v_{\delta_2+1}, \ldots, v_d \in \mathfrak{F}$ be $d - \delta_2$ vectors which together with vectors in (7.2) form a basis for $\mathfrak{F}$. With $\mu_i = 2$ for $i \in \{1, \ldots, \delta_2\}$ and $\mu_i = 1$ for $i \in \{\delta_2 + 1, \ldots, d\}$, this is a basis whose existence is claimed in the theorem.

**3.** Let $m \in \mathbb{N} \setminus \{1\}$ be arbitrary and assume that the theorem holds whenever $S$ is an operator without eigenvalues in a finite-dimensional space such that $\#\delta_S = m$, that is, such that $S^{m-1} \neq 0$ and $S^m = 0$. Let $T$ be an operator without eigenvalues in a finite-dimensional space $\mathfrak{G}$ such that $\#\delta_T = m + 1$, that is $T^m \neq 0$ and $T^{m+1} = 0$. In particular, since $m \in \mathbb{N} \setminus \{1\}$, we have $T^2 \neq 0$. Set

$$\delta_T = (\eta_1, \ldots, \eta_m, \eta_{m+1}), \quad \text{with} \quad \eta_k = \dim T^{k-1} \quad \text{for} \quad k \in \{1, \ldots, m+1\},$$

and $d_T = \eta_1 - \eta_2 = \operatorname{codim}(\operatorname{dom} T)$. Since $m \in \mathbb{N} \setminus \{1\}$ and $\eta_1 > \eta_2 > \cdots > \eta_{m+1} > 0$, see (3.4), we must have $\dim \mathfrak{G} = \eta_1 \geq 3$.

Set $\mathfrak{F} = \operatorname{ran} T$,

$$S = T|_{(\operatorname{dom} T) \cap (\operatorname{ran} T)}$$

and denote by $d_S$ the codimension of $\operatorname{dom} S$ in $\mathfrak{F} = \operatorname{ran} T$. Then $S$ is an operator in $\mathfrak{F}$ without eigenvalues. We set $S^0 = I_{\mathfrak{F}}$ and prove that for all $k \in \mathbb{N}_0$

$$S^k T = T^{k+1} \text{ on } \operatorname{dom} T^{k+1} \text{ and } \operatorname{dom} S^k = T \operatorname{dom} T^{k+1} (= (\operatorname{dom} T^k) \cap (\operatorname{ran} T)). \tag{7.3}$$

The first equality trivially holds for $k = 0$. Assume it holds for some $k \in \mathbb{N}_0$ and let $x \in \operatorname{dom} T^{k+2}$. Then $Tx \in \operatorname{dom} T^{k+1} \subset \operatorname{dom} T$, hence $Tx \in \operatorname{dom} S$ and consequently

$$T^{k+2}x = T^{k+1}Tx = S^k TTx = S^k STx = S^{k+1}Tx, \quad x \in \operatorname{dom} T^{k+2}.$$

As to the second equality in (7.3), by the first equality in (7.3), we have $T \operatorname{dom} T^{k+1} \subset \operatorname{dom} S^k$. We prove the reverse inclusion: $\operatorname{dom} S^k \subset T \operatorname{dom} T^{k+1}$. Since $S^0 = I_{\operatorname{ran} T}$, this

holds for $k = 0$. Assume it holds for $k \in \mathbb{N}_0$ and let $x \in \operatorname{dom} S^{k+1}$. Then $Sx \in \operatorname{dom} S^k \subset T \operatorname{dom} T^{k+1}$. Hence $x \in \operatorname{dom} S \subset \operatorname{ran} T$ and $Tx = Sx = Ty$ for some $y \in \operatorname{dom} T^{k+1}$. It follows that $x = y \in \operatorname{dom} T^{k+1} \cap \operatorname{ran} T = T \operatorname{dom} T^{k+2}$.

The first equality in (7.3) implies $S^{m-1} \neq 0$. The second equality implies that $S^m = 0$. It also implies that $\operatorname{dom} S^{k-1} = \operatorname{dom} T^k$ and therefore, for the numbers

$$\delta_k := \dim S^{k-1}, \qquad k \in \{1, \ldots, m\},$$

we have $\delta_k = \eta_{k+1}$. Furthermore, by (3.5),

$$d_S := \delta_1 - \delta_2 = \eta_2 - \eta_3 \leq \eta_1 - \eta_2 = d_T.$$

Let

$$\mu_S = (\mu_1, \ldots, \mu_{d_S}) = \mathsf{Con}(\mathsf{Der}\, \delta_S) \quad \text{and} \quad \mu_T = (\sigma_1, \ldots, \sigma_{d_T}) = \mathsf{Con}(\mathsf{Der}\, \delta_T).$$

It is easily verified that

$$\sigma_k = \begin{cases} \mu_k + 1 & \text{for all} \quad k \in \{1, \ldots, d_S\}, \\ 1 & \text{for all} \quad k \in \{d_S + 1, \ldots, d_T\}. \end{cases}$$

By the inductive hypothesis applied to the operator $S$ in $\mathfrak{F}$ there exist vectors $w_1, \ldots, w_{d_S}$ in $\mathfrak{F} = \operatorname{ran} T$ such that

$$\left\{ w_j, Sw_j, \ldots, S^{\mu_j - 1} w_j \ : \ j \in \{1, \ldots, d_S\} \right\}, \tag{7.4}$$

is a basis for $\mathfrak{F} = \operatorname{ran} T$. Let $v_1, \ldots, v_{d_S} \in \operatorname{dom} T$ be such that $Tv_j = w_j$ for $j \in \{1, \ldots, d_S\}$. We append the $d_S$ vectors $v_1, \ldots, v_{d_S}$ to the vectors in (7.4) to get the following set of linearly independent vectors in $\mathfrak{G}$:

$$\left\{ v_j, Tv_j, \ldots, T^{\mu_j} v_j \ : \ j \in \{1, \ldots, d_S\} \right\}. \tag{7.5}$$

Here we used that $S^k T = T^{k+1}$ for all $k \in \mathbb{N}$. There are $d_S + \sum \mu$ linearly independent vectors in the set in (7.5). Since

$$\dim \mathfrak{G} = d_T + \dim \mathfrak{F} = d_T + \sum \mu,$$

we need exactly $d_T - d_S$ more linearly independent vectors to get a basis for $\mathfrak{G}$.

Let $v_{d_S+1}, \ldots, v_{d_T}$ be linearly independent vectors in $\mathfrak{G}$ which, when appended to the vectors in (7.5), form a basis for $\mathfrak{G}$. Then the set

$$\left\{ v_j, Tv_j, \ldots, T^{\sigma_j - 1} v_j \ : \ j \in \{1, \ldots, d_T\} \right\}$$

is a basis for $\mathfrak{G}$. This completes the proof of (I).

**4.** To prove (II) notice that $\Phi$ maps the basis in (7.1) onto the standard basis of the canonical subspace $\mathfrak{C}_\mu$. Therefore $\Phi$ extends to a linear bijection. The second claim is straightforward since the property $\Phi S = S_{\mathfrak{C}_\mu} \Phi$ is easily verified on the basis vectors.   □

**Corollary 7.2.** *Consider the set of all ordered pairs $\{\mathfrak{F}, S\}$ in which $\mathfrak{F}$ is a finite-dimensional vector space and $S$ is an operator without eigenvalues in $\mathfrak{F}$. Introduce the equivalence relation on this set by setting*

$$\{\mathfrak{F}_1, S_1\} \sim \{\mathfrak{F}_2, S_2\} \quad \Leftrightarrow \quad \exists \, \Phi : \mathfrak{F}_1 \to \mathfrak{F}_2 \text{ linear bijection such that } \Phi S_1 = S_2 \Phi.$$

*Then in each equivalence class there exists a unique pair consisting of a canonical space of vector polynomials and its operator of multiplication by the independent variable. There are exactly $\mathsf{p}(n)$ equivalence classes in which $\dim \mathfrak{F} = n$.*

## 8. Two related nilpotent operators

Let $n \in \mathbb{N}$. Here we introduce the operation of degree truncation of a scalar polynomial. Notice that for each scalar polynomial $p(z) \in \mathbb{C}[z]$ there exist unique polynomials $q(z) \in \mathbb{C}[z]_{<n}$ and $r(z) \in \mathbb{C}[z]$ such that

$$p(z) = q(z) + z^n r(z).$$

The polynomial $q(z)$ is the truncation of the polynomial $p(z)$ to degree $< n$; we denote it by $\lfloor p(z) \rfloor_{<n}$.

Next we introduce the truncation operator for vector polynomials. Let $\mathfrak{C}_\mu \subset \mathbb{C}^d[z]$ with $\mu = (\mu_1, \ldots, \mu_d)$ be a canonical subspace of vector polynomials. We define the canonical projection $T_{\mathfrak{C}_\mu} : \mathbb{C}^d[z] \to \mathfrak{C}_\mu$ by setting

$$(T_{\mathfrak{C}_\mu} f)(z) = \left[ \lfloor f_1(z) \rfloor_{<\mu_1} \quad \cdots \quad \lfloor f_d(z) \rfloor_{<\mu_d} \right]^\top$$

for an arbitrary $f(z) = \begin{bmatrix} f_1(z) & \cdots & f_d(z) \end{bmatrix}^\top \in \mathbb{C}^d[z]$. Further, we define the *nilpotent extension* $N_{\mathfrak{C}_\mu} : \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ of $S_{\mathfrak{C}_\mu}$ by setting

$$(N_{\mathfrak{C}_\mu} f)(z) = \left[ \lfloor z f_1(z) \rfloor_{<\mu_1} \quad \cdots \quad \lfloor z f_d(z) \rfloor_{<\mu_d} \right]^\top$$

for an arbitrary $f(z) = \begin{bmatrix} f_1(z) & \cdots & f_d(z) \end{bmatrix}^\top \in \mathfrak{C}_\mu$, or, equivalently, by

$$N_{\mathfrak{C}_\mu} = T_{\mathfrak{C}_\mu} S_{\mathbb{C}^d[z]} \big|_{\mathfrak{C}_\mu}.$$

Another important nilpotent operator on a canonical space of vector polynomials $\mathfrak{C}_\mu$ is the differentiation operator $D_{\mathfrak{C}_\mu} : \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ defined by

$$(D_{\mathfrak{C}_\mu} f)(z) = \left[ f_1'(z) \; \cdots \; f_d'(z) \right]^\top \quad \text{where} \quad f(z) = \left[ f_1(z) \; \cdots \; f_d(z) \right]^\top \in \mathfrak{C}_\mu.$$

Here $g'(z)$ denotes the derivative of $g(z) \in \mathbb{C}[z]$.

Next we restate the classical Jordan canonical form theorem, see [17, § 58, Theorem 2] or [28, Section 4.3] for everywhere defined operators in the spirit of Theorem 7.1.

**Theorem 8.1.** *Let $\mathfrak{F}$ be a finite-dimensional vector space and let $A$ be an operator defined on all of $\mathfrak{F}$. Let $A = L + N$ be a Jordan decomposition of $A$ where $L$ is diagonalizable, $N$ is nilpotent and both commute with $A$. Let $m$ be the nilpotency index of $N$ and let $\delta_N = (\delta_1, \ldots, \delta_m)$ be the tuple of nonzero dimensions of the ranges of powers of $N$:*

$$\delta_k := \dim(\operatorname{ran} N^{k-1}) \quad \text{for} \quad k \in \{1, \ldots, m\}.$$

*Let $\mu = (\mu_1, \ldots, \mu_d) = \mathsf{Con}(\mathsf{Der}\,\delta_N)$. The following statements hold.*

(I) *We have $d = \dim(\ker N)$ and there exist vectors $w_1, \ldots, w_d$ in $\mathfrak{F}$ such that:*

    (i) *the set $\left\{ N^{\mu_1 - 1} w_1, \cdots, N^{\mu_d - 1} w_d \right\}$ is a basis for $\ker N$,*

    (ii) *the set $\left\{ w_l, \ldots, N^{\mu_l - 1} w_l : l \in \{1, \ldots, d\} \right\}$ is a basis for $\mathfrak{F}$,*

    (iii) *there exist $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ such that*

$$A N^{k-1} w_l = \lambda_l N^{k-1} w_l + N^k w_l \quad \text{for all } k \in \{1, \ldots, \mu_l\} \text{ and all } l \in \{1, \ldots, d\}.$$

(II) *The function defined by*

$$\Phi N^{k-1} w_l = e_{d,l} z^{k-1} \quad \text{for all } k \in \{1, \ldots, \mu_l\} \text{ and all } l \in \{1, \ldots, d\}$$

*extends linearly to a linear bijection $\Phi : \mathfrak{F} \to \mathfrak{C}_\mu$ such that*

$$\Phi A = \left( \operatorname{Diag}(\lambda_1, \ldots, \lambda_d) + N_{\mathfrak{C}_\mu} \right) \Phi.$$

(III) *The function defined by*

$$\Psi N^{k-1} w_l = \frac{1}{(\mu_l - k)!} e_{d,l} z^{\mu_l - k} \quad \text{for all} \quad k \in \{1, \ldots, \mu_l\} \text{ and all } l \in \{1, \ldots, d\}$$

*extends linearly to a linear bijection $\Psi : \mathfrak{F} \to \mathfrak{C}_\mu$ such that*

$$\Psi A = \left( \operatorname{Diag}(\lambda_1, \ldots, \lambda_d) + D_{\mathfrak{C}_\mu} \right) \Psi.$$

Notice that with $L = 0$ in Theorem 8.1 parts (II) and (III) establish similarity between the nilpotent operators $N_{\mathfrak{C}_\mu}$ and $D_{\mathfrak{C}_\mu}$. More generally, Theorem 8.1 with $L = 0$ yields the following corollary.

**Corollary 8.2.** *Let $\mathfrak{E}$ and $\mathfrak{F}$ be finite-dimensional vector spaces and let $M : \mathfrak{E} \to \mathfrak{E}$ and $N : \mathfrak{F} \to \mathfrak{F}$ be nilpotent operators. The operators $M$ and $N$ are similar if and only if*

$$\dim(\operatorname{ran} M^{k-1}) = \dim(\operatorname{ran} N^{k-1}) \quad \text{for all} \quad k \in \mathbb{N}.$$

**Theorem 8.3.** *Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator in $\mathfrak{F}$ without eigenvalues. There exists a nilpotent operator $N$ on $\mathfrak{F}$ such that $N|_{\operatorname{dom} S} = S$, $\dim(\ker N) = \operatorname{codim}(\operatorname{dom} S)$ and $\operatorname{ran} S^k = \operatorname{ran} N^k$ for all $k \in \mathbb{N}$. Such an operator $N$ is unique up to similarity.*

*Conversely, if $N$ is a nilpotent operator on $\mathfrak{F}$, then there exists an operator $S$ in $\mathfrak{F}$ without eigenvalues such that $N|_{\operatorname{dom} S} = S$, $\dim(\ker N) = \operatorname{codim}(\operatorname{dom} S)$ and $\operatorname{ran} S^k = \operatorname{ran} N^k$ for all $k \in \mathbb{N}$. Such an operator $S$ is unique up to similarity.*

**Proof.** By Theorem 7.1 there exist a canonical space $\mathfrak{C}_\mu$ of vector polynomials and a linear bijection $\Phi : \mathfrak{F} \to \mathfrak{C}_\mu$ such that $\Phi S = S_{\mathfrak{C}_\mu} \Phi$. The nilpotent operator $N$ defined by

$$N = \Phi^{-1} N_{\mathfrak{C}_\mu} \Phi$$

has all the properties stated in the theorem. The uniqueness claim follows from Corollary 8.2.

Let $N$ be a nilpotent operator on $\mathfrak{F}$. In the notation of Theorem 8.1 with $A = N$, let $S$ be the restriction of $N$ onto the subspace spanned by the vectors:

$$\left\{ w_l, \ldots, N^{\mu_l - 1} w_l \,:\, l \in \{1, \ldots, d\} \right\} \setminus \left\{ N^{\mu_1 - 1} w_1, \cdots, N^{\mu_d - 1} w_d \right\}.$$

Since this span does not contain any vectors from $\ker N$, the operator $S$ does not have eigenvalues. The other claims about $S$ are easily verified. The uniqueness claim follows from Theorem 7.1 and Corollary 7.2.  $\square$

**Theorem 8.4.** *Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator in $\mathfrak{F}$ without eigenvalues. There exists a nilpotent operator $D$ on $\mathfrak{F}$ such that $DS - SD = I$ holds on $\operatorname{dom} S$ and $\operatorname{ran} D^k = \operatorname{dom} S^k$ for all $k \in \mathbb{N}$. Such an operator $D$ is unique up to similarity.*

*Conversely, if $D$ is a nilpotent operator on $\mathfrak{F}$, then there exists an operator $S$ in $\mathfrak{F}$ without eigenvalues such that $DS - SD = I$ and $\operatorname{ran} D^k = \operatorname{dom} S^k$ for all $k \in \mathbb{N}$. Such an operator $S$ is unique up to similarity.*

**Proof.** By Theorem 7.1 there exist a canonical space $\mathfrak{C}_\mu$ of vector polynomials and a linear bijection $\Phi : \mathfrak{F} \to \mathfrak{C}_\mu$ such that $\Phi S = S_{\mathfrak{C}_\mu} \Phi$. Since $D_{\mathfrak{C}_\mu} S_{\mathfrak{C}_\mu} - S_{\mathfrak{C}_\mu} D_{\mathfrak{C}_\mu} = I_{\mathfrak{C}_\mu}$, the nilpotent operator $D$ defined by

$$D = \Phi^{-1} D_{\mathfrak{C}_\mu} \Phi$$

has all the properties stated in the theorem. The uniqueness claim follows from Corollary 8.2.

Let $D$ be a nilpotent operator on $\mathfrak{F}$. Let $\Psi : \mathfrak{F} \to \mathfrak{C}_\mu$ be the linear bijection from Theorem 8.1(III) with $A = N$. Then the operator

$$S = \Psi^{-1} S_{\mathfrak{C}_\mu} \Psi$$

has all the properties stated in the theorem. The uniqueness claim follows from Theorem 7.1 and Corollary 7.2. $\quad\square$

**Remark 8.5.** Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator in $\mathfrak{F}$ without eigenvalues. Let $N$ be a nilpotent operator whose existence was established in Theorem 8.3 and let $D$ be a nilpotent operator whose existence was established in Theorem 8.4. Corollary 8.2 implies that the operators $N$ and $D$ are similar.

**Remark 8.6.** The *Weyr characteristics* of any nilpotent operator $N$ is defined in [27] as the tuple of positive integers from the sequence

$$\dim\big(\operatorname{nul} N^k\big) - \dim\big(\operatorname{nul} N^{k-1}\big) = \dim\big(\operatorname{ran} N^{k-1}\big) - \dim\big(\operatorname{ran} N^k\big), \quad k \in \mathbb{N}.$$

For $N$ and $S$ in Theorem 8.3 this definition coincides with our definition of $\nu_S$. The *Segre characteristics* of $N$ is defined in [27] as the conjugate tuple to its Weyr characteristic; hence it coincides with $\mu_S = \operatorname{Con} \nu_S$. The same holds for $D$ and $S$ in Theorem 8.4.

**Example 8.7.** The following nilpotent operator given by the $16 \times 16$ matrix in the Weyr canonical form was studied in [27]:

$$
N = \left[\begin{array}{ccccccc|ccccc|cc|cc}
0 & & & & & & & 1 & & & & & & & & \\
 & 0 & & & & & & & 1 & & & & & & & \\
 & & 0 & & & & & & & 1 & & & & & & \\
 & & & 0 & & & & & & & 1 & & & & & \\
 & & & & 0 & & & & & & & 1 & & & & \\
 & & & & & 0 & & 0 & 0 & 0 & 0 & 0 & & & & \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
 & & & & & & & 0 & & & & & 1 & & & \\
 & & & & & & & & 0 & & & & & 1 & & \\
 & & & & & & & & & 0 & & & 0 & 0 & & \\
 & & & & & & & & & & 0 & & 0 & 0 & & \\
 & & & & & & & & & & & 0 & 0 & 0 & & \\
\hline
 & & & & & & & & & & & & 0 & & 1 & \\
 & & & & & & & & & & & & & 0 & & 1 \\
\hline
 & & & & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & & & & 0 \\
\end{array}\right].
$$

By [27] $N$ has Weyr characteristic $(7, 5, 2, 2)$ and Segre characteristic $(4, 4, 2, 2, 2, 1, 1)$.

Denote by $S$ the restriction of $N$ onto

$$\operatorname{dom} S = \operatorname{span}\{e_{16,8}, \ldots, e_{16,16}\},$$

as in Theorem 8.3. Since the null space of $N$ is

$$\text{nul } N = \text{span}\{e_{16,1}, \dots, e_{16,7}\},$$

the operator $S$ does not have eigenvalues. We have

$$\text{nul } N^2 = \text{span}\{e_{16,1}, \dots, e_{16,12}\}$$

and $S^2$ is the restriction of $N^2$ to

$$\text{dom } S^2 = \text{span}\{e_{16,13}, e_{16,14}, e_{16,15}, e_{16,16}\}.$$

Further,

$$\text{nul } N^3 = \text{span}\{e_{16,1}, \dots, e_{16,14}\},$$

$S^3$ is the restriction of $N^3$ to

$$\text{dom } S^3 = \text{span}\{e_{16,15}, e_{16,16}\}$$

and $S^4 = 0$ and $N^4 = 0$. Therefore,

$$\text{ran } S^k = \text{ran } N^k \quad \text{for all} \quad k \in \{0, 1, 2, 3, 4\}.$$

Thus $\delta_S = (16, 9, 4, 2)$ and hence

$$\nu_S = \text{Der } \delta_S = (7, 5, 2, 2), \quad \mu_S = \text{Con } \nu_S = (4, 4, 2, 2, 2, 1).$$

As stated in Remark 8.6, $\nu_S$ coincides with the Weyr characteristic and $\mu_S$ coincides with the Segre characteristic of $N$.

The matrix

$$D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

is a nilpotent operator satisfying Theorem 8.4 with $S$ as above. This can be verified directly or using Theorem 7.1 as in the proof of Theorem 8.4.

## 9. Shifts and symmetrizability of operators without eigenvalues

Theorem 7.1 (I) brings forward a special class of operators without eigenvalues.

**Definition 9.1.** Let $\mathfrak{F}$ be a vector space of dimension $n$. An operator $S$ in $\mathfrak{F}$ will be called a *shift* if there exists a basis $\{v_1, \ldots, v_n\}$ of $\mathfrak{F}$ such that $\mathrm{dom}\, S = \mathrm{span}\{v_1, \ldots, v_{n-1}\}$ and $Sv_k = v_{k+1}$ for all $k \in \{1, \ldots, n-1\}$; if $n = 1$, then $S = \{\{0, 0\}\}$.

In [14, Definition 5.2] a shift operator is any extension of a shift in Definition 9.1 to all of $\mathfrak{F}$. Now Theorem 7.1 (I) can be restated as follows.

**Corollary 9.2.** *Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator in $\mathfrak{F}$ without eigenvalues. Let $\delta_S$ be the tuple of dimensions of $S$ and set $l = \delta_2 - \delta_3 = \mathrm{dim}(\mathrm{ran}\, S) - \mathrm{dim}(\mathrm{ran}\, S^2)$ and $\mu = (\mu_1, \ldots, \mu_d) = \mathsf{Con}(\mathsf{Der}\, \delta_S)$. Then there exist subspaces $\mathfrak{F}_0, \mathfrak{F}_1, \ldots, \mathfrak{F}_l$ of $\mathfrak{F}$ and shifts $S_k$ in $\mathfrak{F}_k$ with $k \in \{1 \ldots, l\}$ such that*

$$\mathrm{dim}\, \mathfrak{F}_0 = d - l, \quad \mathrm{dim}\, \mathfrak{F}_j = \mu_j, \quad j \in \{1, \ldots, l\},$$

$$\mathfrak{F} = \mathfrak{F}_0 \dot{+} \mathfrak{F}_1 \dot{+} \cdots \dot{+} \mathfrak{F}_l \quad and \quad S = S_1 \dot{+} \cdots \dot{+} S_l.$$

Next we will study inner products on a finite-dimensional vector space in which a given operator without eigenvalues is symmetric.

For an operator $A$ defined on all of a finite-dimensional vector space $\mathfrak{F}$ Theorem 5.1.1 and Corollary 5.1.2 in [16] describe all the inner products with respect to which $A$ is self-adjoint. It turns out that the Jordan structure of $A$ restricts the possible numbers of positive and negative squares of such an inner product. For example, if $n = \mathrm{dim}\, \mathfrak{F}$ and if $A$ is a nilpotent operator whose index of nilpotency is $n$, then each inner product on $\mathfrak{F}$ with respect to which $A$ is self-adjoint must have $\lfloor n/2 \rfloor$ positive and $\lceil n/2 \rceil$ negative squares or $\lceil n/2 \rceil$ positive and $\lfloor n/2 \rfloor$ negative squares; to some extent the inner product is uniquely determined. On the other extreme, if $A$ is diagonalizable and if its spectrum is real, then there are no restrictions on the numbers of positive and negative squares for an inner product with respect to which $A$ is self-adjoint. In the next theorem we show that no such restrictions exist for an operator without eigenvalues.

**Theorem 9.3.** *Let $\mathfrak{F}$ be a finite-dimensional vector space of dimension $n$ and let $p$ and $q$ be nonnegative integers such that $n = p + q$. Let $S$ be an operator without eigenvalues in $\mathfrak{F}$. Then there exists an inner product $[\cdot, \cdot]$ on $\mathfrak{F}$ with $p$ positive and $q$ negative squares such that $S$ is symmetric in the Pontryagin space $(\mathfrak{F}, [\cdot, \cdot])$.*

**Proof.** We will first prove the theorem for $S$ being a shift. Let $\{v_1, \ldots, v_n\}$ be the basis for $\mathfrak{F}$ such that $Sv_j = v_{j+1}$ for $j \in \{1, \ldots, n-1\}$. Let $[\cdot, \cdot]$ be an inner product on $\mathfrak{F}$ and denote by $G$ the Gram matrix of $[\cdot, \cdot]$ with respect to the basis of $\{v_1, \ldots, v_n\}$.

Recall that $G$ is the $n \times n$ self-adjoint matrix whose entries $G_{jk}$ are defined by

$$G_{jk} = \big[v_k, v_j\big], \quad j, k \in \{1, \ldots, n\}. \tag{9.1}$$

The operator $S$ is symmetric in $\big(\mathfrak{F}, [\,\cdot\,,\,\cdot\,]\big)$ if and only if for all $j, k \in \{1, \ldots, n-1\}$ we have

$$G_{j,k+1} = [Sv_k, v_j] = \big[v_k, Sv_j\big] = G_{j+1,k}.$$

The preceding $(n-1)^2$ equalities involving the entries of $G$ are equivalent to the following $(n-1)^2$ equalities

$$G_{1,k} = G_{1+i,k-i}, \quad i \in \{1, \ldots, k-1\}, \ k \in \{2, \ldots, n\},$$
$$G_{k,n} = G_{k+i,n-i}, \quad i \in \{1, \ldots, n-k\}, \ k \in \{2, \ldots, n-1\}.$$

Since $G$ is a self-adjoint matrix, the last $(n-1)^2$ equalities hold if and only if there exist $h_1, \ldots, h_{2n-1} \in \mathbb{R}$ such that

$$G_{j,k} = h_{j+k-1} \quad \text{for all} \quad j, k \in \{1, \ldots, n\}.$$

This proves that $S$ is symmetric in $\big(\mathfrak{F}, [\,\cdot\,,\,\cdot\,]\big)$ if and only if the Gram matrix of $[\,\cdot\,,\,\cdot\,]$ with respect to the basis of $\{v_1, \ldots, v_n\}$ is an invertible real Hankel matrix.

Direct calculations show that for an arbitrary invertible real Vandermonde matrix $V$ and an arbitrary invertible real diagonal matrix $D$ the matrix $VDV^\top$ is an invertible real Hankel matrix. This is the easy direction of the Vandermonde factorization theorem for Hankel matrices, see [12, Theorem 7.9], [18, Corollary I.2.8], [9]. Define the inner product on $\mathfrak{F}$ by (9.1) where $G = VDV^\top$ with $D$ having $p$ positive and $q$ negative diagonal entries and with an invertible real Vandermonde matrix $V$ to complete the proof of the theorem for a shift.

The general statement follows from Corollary 9.2.   □

**Remark 9.4.** Let $\mathfrak{F}$ be a finite-dimensional vector space and let $S$ be an operator without eigenvalues in $\mathfrak{F}$. The nilpotent operator $N$ studied in Theorem 8.3 is an extension of $S$ onto the entire $\mathfrak{F}$. Therefore $S$ is symmetric with respect to each inner product on $\mathfrak{F}$ with respect to which $N$ is self-adjoint. All such inner products are described in Theorem 5.1.1 and Corollary 5.1.2 in [16].

The situation with the operator $D$ studied in Theorem 8.4 is different. To clarify this, let $S$ and $D$ be as in Theorem 8.4 and notice that we have that $S^2 = \{\{0,0\}\}$ if and only if $D^2 = 0$. We will show that there exists an inner product on $\mathfrak{F}$ with respect to which $S$ is symmetric and $D$ is self-adjoint if and only if $D^2 = 0$. Assume that $[\,\cdot\,,\,\cdot\,]$ is an inner product on $\mathfrak{F}$ with respect to which $S$ is symmetric and $D$ is self-adjoint. Then, using $\operatorname{dom} S = \operatorname{ran} D$ and $DS - SD = I$ on $\operatorname{dom} S$, we have for all $x, y \in \operatorname{dom} S$

$$[DSx, y] = [x, y] + [SDx, y] = [x, y] + [x, DSy] = 2[x, y] + [x, SDy] = 2[x, y] + [DSx, y],$$

that is $[x, y] = 0$. Hence, $\operatorname{dom} S = \operatorname{ran} D$ is a neutral subspace of $(\mathfrak{F}, [\,\cdot\,,\cdot\,])$. Consequently, for $x, y \in \mathfrak{F}$ we have

$$[D^2 x, y] = [Dx, Dy] = 0,$$

implying that $D^2 = 0$.

To prove the converse, assume that $S$ and $D$ have all the properties in Theorem 8.4 and, in addition, $D^2 = 0$, or, equivalently, $S^2 = \{\{0, 0\}\}$. These assumptions imply that $(\ker D) \cap (\operatorname{ran} S) = \{0\}$, $\operatorname{dom} S = \operatorname{ran} D \subseteq \ker D$ and $D(x - SDx) = 0$, $x \in \mathfrak{F}$. Therefore, $\mathfrak{F} = (\ker D) \dotplus (\operatorname{ran} S)$ and $DSx = x$ for all $x \in \operatorname{dom} S$. Let $\langle\,\cdot\,,\cdot\,\rangle$ be an arbitrary positive definite inner product on $\ker D$ and let $\mathfrak{F}_0$ be the orthogonal complement of $\operatorname{dom} S$ in this inner product. Then

$$\mathfrak{F} = \mathfrak{F}_0 \dotplus (\operatorname{dom} S) \dotplus (\operatorname{ran} S).$$

For arbitrary $x, y \in \mathfrak{F}$ define $[\,\cdot\,,\cdot\,]$ by

$$[x, y] = \langle x_0, y_0 \rangle + \langle x_2, y_1 \rangle + \langle x_1, y_2 \rangle,$$

where

$$x = x_0 + x_1 + Sx_2 \quad \text{with} \quad x_0 \in \mathfrak{F}_0, \quad x_1, x_2 \in \operatorname{dom} S,$$
$$y = y_0 + y_1 + Sy_2 \quad \text{with} \quad y_0 \in \mathfrak{F}_0, \quad y_1, y_2 \in \operatorname{dom} S.$$

Clearly $[\,\cdot\,,\cdot\,]$ is an indefinite inner product on $\mathfrak{F}$ in which $\operatorname{dom} S$ and $\operatorname{ran} S$ are neutral subspaces. With $x, y \in \mathfrak{F}$ as above, since $DS$ acts as an identity on $\operatorname{dom} S$ we have $Dx = x_2$ and $Dy = y_2$. Therefore, for all $x, y \in \mathfrak{F}$,

$$[Dx, y] = [x_2, y] = \langle x_2, y_2 \rangle = [x, y_2] = [x, Dy],$$

proving that $D$ is self-adjoint in $(\mathfrak{F}, [\,\cdot\,,\cdot\,])$. Similarly, for all $x_1, y_1 \in \operatorname{dom} S$,

$$[Sx_1, y_1] = \langle x_1, y_1 \rangle = [x_1, Sy_1],$$

proving that $S$ is symmetric.

## 10. Canonical subspaces of $\mathbb{C}^d[z]$ as reproducing kernel spaces

The next theorem was implicitly proved in [10]. Here we give a simpler proof.

**Lemma 10.1.** *Let* $(\mathfrak{K}, [\,\cdot\,,\cdot\,]_{\mathfrak{K}})$ *be a Pontryagin space with positive and negative index equal to* $n$. *Let* $\mathfrak{N}$ *be a neutral subspace of* $\mathfrak{K}$ *with* $\dim \mathfrak{N} = \tau < n$ *and denote by* $\mathfrak{N}^{[\perp]}$ *the orthogonal complement of* $\mathfrak{N}$ *in* $(\mathfrak{K}, [\,\cdot\,,\cdot\,]_{\mathfrak{K}})$. *Then the quotient space* $\mathfrak{N}^{[\perp]}/\mathfrak{N}$ *with the induced inner product is a Pontryagin space with positive and negative index equal to* $n - \tau$.

**Proof.** Let $\mathfrak{N}^{[\perp]} = \mathfrak{N} + \mathfrak{N}_- + \mathfrak{N}_+$ be a fundamental decomposition of $\mathfrak{N}^{[\perp]}$, see [6, Sections I.4 and I.11]. Since $\mathfrak{N}$ is a neutral subspace of $\mathfrak{K}$, we have $n \geq \tau + \dim \mathfrak{N}_\pm$. Since also

$$2n - \tau = \dim \mathfrak{N}^{[\perp]} = \tau + \dim \mathfrak{N}_- + \dim \mathfrak{N}_+,$$

we conclude $\dim \mathfrak{N}_- = \dim \mathfrak{N}_+ = n - \tau$.  $\square$

The next lemma is [10, Lemma 2.7].

**Lemma 10.2.** *Let* $d, p \in \mathbb{N}$ *and let* $K(z, w)$ *be a Hermitian* $d \times d$ *matrix polynomial kernel of degree* $p - 1$. *For* $q \in \mathbb{N}$ *set*

$$L_q(z, w) = -\mathrm{i}\,(z^q - w^{*q})K(z, w), \qquad z, w \in \mathbb{C}.$$

*If* $q \geq p$, *then the positive and the negative index of the reproducing kernel Pontryagin space with kernel* $L_q(z, w)$ *are equal and coincide with the dimension of the reproducing kernel Pontryagin space with kernel* $K(z, w)$.

**Theorem 10.3.** *Let* $d \in \mathbb{N}$, *let* $\mathsf{Q}$ *be a* $2d \times 2d$ *self-adjoint matrix with* $d$ *positive and* $d$ *negative eigenvalues and let* $\mathscr{P}(z)$ *be a* $d \times 2d$ *matrix polynomial whose row degrees are* $\mu_1, \ldots, \mu_d$. *Assume that* $\mathscr{P}(z)$ *has the following properties:*

(a)  $\mathscr{P}(z)\mathsf{Q}^{-1}\mathscr{P}(z^*)^* = 0$ *for all* $z \in \mathbb{C}$.
(b)  $\operatorname{rank}\mathscr{P}(z) = d$ *for all* $z \in \mathbb{C}$.
(c)  $\operatorname{rank} P_\infty = d$.
(d)  $\mu_1 \geq \cdots \geq \mu_d \geq 1$.

*Then the Pontryagin space with reproducing polynomial Nevanlinna kernel*

$$K_{\mathscr{P}}(z, w) = \frac{\mathrm{i}}{z - w^*}\mathscr{P}(z)\mathsf{Q}^{-1}\mathscr{P}(w)^*, \quad z \neq w^*, \quad z, w \in \mathbb{C}, \tag{10.1}$$

*is the canonical subspace* $\mathfrak{C}_\mu$ *of* $\mathbb{C}^d[z]$ *where* $\mu = (\mu_1, \ldots, \mu_d)$ *and the operator* $S_{\mathfrak{C}_\mu}$ *is symmetric in this Pontryagin space.*

**Proof. 1.** Set $p = \mu_1$ and consider the space $\mathbb{C}^{2d}[z]_{<p}$ with the inner product

$$[f, g]_{\mathsf{Q}} := \sum_{j=0}^{p-1} b_{p-1-j}^* \mathsf{Q}^{-1} a_j \tag{10.2}$$

where

$$f(z) = \sum_{j=0}^{p-1} a_j z^j, \quad g(z) = \sum_{j=0}^{p-1} b_j z^j, \quad a_j, b_j \in \mathbb{C}^{2d}.$$

In this Pontryagin space we define a special subspace related to the $d \times 2d$ matrix polynomial $\mathscr{P}(z)$:

$$\mathfrak{L} := \operatorname{span}\left\{ \sum_{k=0}^{p-1} z^{p-1-k} w^{*k} \mathscr{P}(w)^* x \ : \ w \in \mathbb{C}, \ x \in \mathbb{C}^d \right\}.$$

For an element $f(z) = \sum_{j=0}^{p-1} a_j z^j \in \mathbb{C}^{2d}[z]_{<p}$ the following equivalences hold:

$$f(z) \in \mathfrak{L}^\perp \Leftrightarrow \left( \sum_{k=0}^{p-1} w^{*k} a_k^* \right) \mathsf{Q}^{-1} \mathscr{P}(w)^* = 0 \quad \forall\, w \in \mathbb{C},$$

$$\Leftrightarrow \mathscr{P}(z) \mathsf{Q}^{-1} f(z) = 0 \quad \forall\, z \in \mathbb{C}, \tag{10.3}$$

$$\Leftrightarrow \forall\, z \in \mathbb{C} \ \exists\, u_z \in \mathbb{C}^d \ \text{such that} \ f(z) = \mathscr{P}(z^*)^* u_z.$$

The last equivalence follows from (a) and (b). To prove that the vector $u_z$ depends polynomially on $z$ we use that the Smith normal form of $\mathscr{P}(z)$ is given by: $\mathscr{P}(z) = \mathscr{U}(z)\begin{bmatrix} I_d & 0 \end{bmatrix}\mathscr{V}(z)$, where $\mathscr{U}(z)$ and $\mathscr{V}(z)$ are unimodular matrices. Then

$$f(z) = \mathscr{P}(z^*)^* u_z = \mathscr{V}(z^*)^* \begin{bmatrix} I_d \\ 0 \end{bmatrix} \mathscr{U}(z^*)^* u_z.$$

Therefore

$$u_z = \mathscr{U}(z^*)^{-*} \begin{bmatrix} I_d & 0 \end{bmatrix} \mathscr{V}(z^*)^{-*} f(z)$$

and the right-hand side belongs to $\mathbb{C}^d[z]$. Hence

$$\mathfrak{L}^\perp = \left\{ f(z) \in \mathbb{C}^{2d}[z]_{<p} \ : \ f(z) = \mathscr{P}(z^*)^* u(z) \ \text{ with } \ u(z) \in \mathbb{C}^d[z] \right\}.$$

**2.** Since $\mathscr{P}(z^*)^*$ has full rank for every $z \in \mathbb{C}$, it acts as an injection on $\mathbb{C}^d[z]$. Therefore

$$\dim \mathfrak{L}^{\perp} = \dim\{u(z) \in \mathbb{C}^d[z] \, : \, \deg(\mathscr{P}(z^*)^* u(z)) < p\}. \tag{10.4}$$

The number on the right-hand side in (10.4) can be expressed in terms of row degrees $\mu_1, \ldots, \mu_d$ of $\mathscr{P}(z)$. By Theorem 4.1, since $\operatorname{rank} P_\infty = d$, the polynomial $\mathscr{P}(z)$ has the predictable degree property (which we state here in transposed form):

$$\deg(\mathscr{P}(z^*)^* u(z)) = \max\{\mu_j + \deg u_j(z) \, : \, j \in \{1, \ldots, d\}\}.$$

Consequently, the space on the right-hand side of the equation in (10.4) equals

$$\mathfrak{M} = \{u(z) \in \mathbb{C}^d[z] \, : \, \deg u_j(z) < p - \mu_j, \ j \in \{1, \ldots, d\}\}.$$

In fact, since $p - \mu_d \geq \cdots \geq p - \mu_1$, $\mathfrak{M} = Z_d \mathfrak{C}_{(p-\mu_d, \ldots, p-\mu_1)}$, where $Z_d$ is the $d \times d$ reverse identity matrix. The dimension of this space is $dp - (\mu_1 + \cdots + \mu_d)$. Hence,

$$\dim \mathfrak{L}^{\perp} = dp - (\mu_1 + \cdots + \mu_d)$$

and

$$\dim \mathfrak{L} = \dim(\mathbb{C}^{2d}[z]_{<p}) - \dim \mathfrak{L}^{\perp} = dp + (\mu_1 + \cdots + \mu_d).$$

**3.** To prove that $\mathfrak{L}^{\perp}$ is a neutral subspace of $(\mathbb{C}^{2d}[z]_{<p}, [\cdot, \cdot]_{\mathsf{Q}})$ we rewrite $\mathfrak{L}^{\perp}$ as

$$\mathfrak{L}^{\perp} = \operatorname{span}\left\{ z^k \mathscr{P}(z^*)^* a \, : \, \begin{matrix} a \in \mathbb{C}^d, \quad k \in \{0, 1, \ldots, p-1\} \\ \text{such that} \ \ z^k \mathscr{P}(z^*)^* a \in \mathbb{C}^{2d}[z]_{<p} \end{matrix} \right\} \tag{10.5}$$

and set $\mathscr{P}(z) = P_0 + z P_1 + \cdots + z^p P_p$ and $P_i = 0$ for negative integers $i$ and for integers $i > p$. Further, let $a, b \in \mathbb{C}^d$ and $k, l \in \{0, 1, \ldots, p-1\}$ be such that $z^k \mathscr{P}(z^*)^* a \in \mathbb{C}^{2d}[z]_{<p}$ and $z^l \mathscr{P}(z^*)^* b \in \mathbb{C}^{2d}[z]_{<p}$. Then $P_j^* a = 0$ for $j \geq p - k$, $P_j^* b = 0$ for $j \geq p - l$ and

$$z^k \mathscr{P}(z^*)^* a = \sum_{j=0}^{p-1} z^j P_{j-k}^* a \quad \text{and} \quad z^l \mathscr{P}(z^*)^* b = \sum_{j=0}^{p-1} z^j P_{j-l}^* b.$$

Therefore

$$\left[z^k \mathscr{P}(z^*)^* a, z^l \mathscr{P}(z^*)^* b\right]_{\mathsf{Q}} = \sum_{j=0}^{p-1} b^* P_{p-1-j+l} \mathsf{Q}^{-1} P_{j-k}^* a$$

$$= \sum_{\substack{r+s=l-k+p-1 \\ r \in \{l,\dots,l+p-1\} \\ s \in \{-k,\dots,-k+p-1\}}} b^* P_r \mathsf{Q}^{-1} P_s^* a$$

$$= \sum_{\substack{r+s=l-k+p-1 \\ r \in \{l,\dots,l+p-1\} \\ s \in \{0,\dots,l-k+p-1\}}} b^* P_r \mathsf{Q}^{-1} P_s^* a \qquad (10.6)$$

$$= \sum_{\substack{r+s=l-k+p-1 \\ r,s \in \{0,\dots,l-k+p-1\}}} b^* P_r \mathsf{Q}^{-1} P_s^* a$$

$$= b^* \left( \sum_{\substack{r+s=l-k+p-1 \\ r,s \in \{0,\dots,l-k+p-1\}}} P_r \mathsf{Q}^{-1} P_s^* \right) a$$

$$= 0,$$

where the last equality holds because the assumption (a) is equivalent to

$$\sum_{\substack{r+s=n \\ r,s \in \{0,\dots,n\}}} P_r \mathsf{Q}^{-1} P_s^* = 0 \quad \text{for all} \quad n \in \mathbb{N}_0,$$

provided $P_s = 0$ for $s > p$. The third equality holds because $P_s = 0$ for negative indices $s$ and $P_s^* a = 0$ for $s \geq p - k$. The fourth equality holds because if $r + s = l - k + p - 1$, then $r \leq l - 1$ implies that $s \geq p - k$ and hence $P_s^* a = 0$ and $r \geq l + p$ implies that $s \leq -1 - k \leq -1$ and hence $P_s = 0$. Now (10.5) and (10.6) yield that $\mathfrak{L}^\perp$ is a neutral subspace of $\left(\mathbb{C}^{2d}[z]_{<p}, [\cdot,\cdot]_{\mathsf{Q}}\right)$.

Lemma 10.1 applied to the Pontryagin space $\left(\mathbb{C}^{2d}[z]_{<p}, [\cdot,\cdot]_{\mathsf{Q}}\right)$ with $n = dp$ and $\tau = dp - (\mu_1 + \cdots + \mu_d)$ implies that

$$\dim\left(\mathfrak{L}/\mathfrak{L}^\perp\right) = 2(\mu_1 + \cdots + \mu_d)$$

and the positive and negative index of $\mathfrak{L}/\mathfrak{L}^\perp$ equal $\mu_1 + \cdots + \mu_d$.

**4.** Denote by $\mathfrak{B}_p$ the reproducing kernel Pontryagin space with kernel

$$L_p(z,w) := -\mathrm{i}\,(z^p - w^{*p}) K_{\mathscr{P}}(z,w) = \left(\sum_{k=0}^{p-1} z^{p-1-k} w^{*k}\right) \mathscr{P}(z) \mathsf{Q}^{-1} \mathscr{P}(w)^*, \quad z, w \in \mathbb{C}.$$

Then

$$\mathfrak{B}_p = \mathrm{span}\left\{\mathscr{P}(z) \mathsf{Q}^{-1} \sum_{k=0}^{p-1} z^{p-1-k} w^{*k} \mathscr{P}(w)^* x \;:\; w \in \mathbb{C}, \; x \in \mathbb{C}^d\right\}$$

and

$$\left[\mathscr{P}(z)\mathsf{Q}^{-1}f, \mathscr{P}(z)\mathsf{Q}^{-1}g\right]_{\mathfrak{B}_p} = \sum_{k=0}^{p-1}\left(v^{*(p-1-k)}\mathscr{P}(v)^*y\right)^*\mathsf{Q}^{-1}\left(w^{*k}\mathscr{P}(w)^*x\right), \qquad (10.7)$$

where

$$f(z) = \sum_{k=0}^{p-1} z^{p-1-k} w^{*k}\mathscr{P}(w)^*x, \quad g(z) = \sum_{k=0}^{p-1} z^{p-1-k} v^{*k}\mathscr{P}(v)^*y.$$

It follows from the definitions of the inner products in (10.7) and (10.2) that the operator of multiplication by $\mathscr{P}(z)\mathsf{Q}^{-1}$ maps $\mathfrak{L} \subset \mathbb{C}^{2d}[z]_{<p}$ isometrically onto $\mathfrak{B}_p$. The second equivalence in (10.3) implies that the null space of this mapping is $\mathfrak{L}^\perp$. Hence, $\dim \mathfrak{B}_p = 2(\mu_1 + \cdots + \mu_d)$ and the positive and the negative index of $\mathfrak{B}_p$ equal $\mu_1 + \cdots + \mu_d$.

**5.** Let $\mathfrak{K}_{\mathscr{P}}$ be the reproducing kernel space with reproducing kernel $K_{\mathscr{P}}(z, w)$. Lemma 10.2 yields $\dim \mathfrak{K}_{\mathscr{P}} = \mu_1 + \cdots + \mu_d$. The space $\mathfrak{K}_{\mathscr{P}}$ is spanned by the columns of the matrices $K_{\mathscr{P}}(z, w)$ with $w \in \mathbb{C}$ and for $j \in \{1, \ldots, d\}$ the degree of the $j$-th row of $K(z, w)$ as a polynomial in $z$ is equal to $\mu_j - 1$. Therefore $\mathfrak{K}_{\mathscr{P}} \subseteq \bigoplus_{j=1}^d \left(\mathbb{C}[z]_{<\mu_j}\right)e_{d,j}$. Since both spaces have dimension $\mu_1 + \cdots + \mu_d$, equality prevails:

$$\mathfrak{K}_{\mathscr{P}} = \bigoplus_{j=1}^d \left(\mathbb{C}[z]_{<\mu_j}\right)e_{d,j} = \mathfrak{C}_{(\mu_1,\ldots,\mu_d)} = \mathfrak{C}_\mu.$$

**6.** It follows from the "only if" part of [10, Theorem 1.1] that $S_{\mathfrak{C}_\mu}$ is symmetric.  □

The next theorem considers reproducing kernel spaces with kernels generated by more general matrix polynomials $\mathscr{T}(z) \in \mathbb{C}^{d \times 2d}[z]$. Part (C) below can be considered as a kind of converse of Theorem 10.3.

**Theorem 10.4.** *Let $d \in \mathbb{N}$, let $\mathsf{Q}$ be a $2d \times 2d$ self-adjoint matrix with $d$ positive and $d$ negative eigenvalues and let $\mathscr{T}(z) \in \mathbb{C}^{d \times 2d}[z]$ be such that:*

(i) $\mathscr{T}(z)\mathsf{Q}^{-1}\mathscr{T}(z^*)^* = 0$ *for all $z \in \mathbb{C}$.*
(ii) $\operatorname{rank}\mathscr{T}(z) = d$ *for some $z \in \mathbb{C}$.*

*Then:*

(A) *There exist $\mathscr{P}(z) \in \mathbb{C}^{d \times 2d}[z]$ which satisfies (a) through (d) in Theorem 10.3 and $\mathscr{W}(z) \in \mathbb{C}^{d \times d}[z]$ with $\det \mathscr{W}(z) \not\equiv 0$ such that*

$$\mathcal{T}(z) = \mathcal{W}(z)\mathcal{P}(z) \quad \text{for all} \quad z \in \mathbb{C}.$$

(B) *Let* $\mathcal{P}(z) \in \mathbb{C}^{d \times 2d}[z]$ *and* $\mathcal{W}(z) \in \mathbb{C}^{d \times d}[z]$ *be as in* (A). *The reproducing kernel Pontryagin space* $\mathfrak{K}_{\mathcal{T}}$ *with reproducing polynomial Nevanlinna kernel*

$$K_{\mathcal{T}}(z,w) := \frac{\mathrm{i}}{z - w^*} \mathcal{T}(z)\mathsf{Q}^{-1}\mathcal{T}(w)^*, \quad z \neq w^*, \quad z, w \in \mathbb{C},$$

*is the subspace of* $\mathbb{C}^d[z]$ *given by:*

$$\mathfrak{K}_{\mathcal{T}} = \big\{ \mathcal{W}(z)f(z) \,:\, f(z) \in \mathfrak{C}_\mu \big\} \quad \text{where} \quad \mu = (\mu_1, \dots, \mu_d).$$

*The operator* $W$ *of multiplication by* $\mathcal{W}(z)$ *is an isometry from* $\mathfrak{K}_{\mathcal{P}} = \mathfrak{C}_\mu$ *onto* $\mathfrak{K}_{\mathcal{T}}$ *and* $WS_{\mathfrak{C}_\mu} = S_{\mathfrak{K}_{\mathcal{T}}}W$. *Moreover, the operator* $S_{\mathfrak{K}_{\mathcal{T}}}$ *is symmetric in the reproducing kernel Pontryagin space* $\mathfrak{K}_{\mathcal{T}}$.

(C) *Let* $\mathcal{P}(z) \in \mathbb{C}^{d \times 2d}[z]$ *and* $\mathcal{W}(z) \in \mathbb{C}^{d \times d}[z]$ *be as in* (A). *The following statements are equivalent:*

   (I) $\mathfrak{K}_{\mathcal{T}} = \mathfrak{C}_\mu$.
   (II) $\mathcal{W}(z) = \big[w_{jk}(z)\big]_{j,k=1}^d$ *is a* $d \times d$ *unimodular matrix such that*

$$\deg w_{jk}(z) \leq \mu_j - \mu_k \quad \text{for all} \quad j, k \in \{1, \dots, d\}. \tag{10.8}$$

   (III) *The row degrees of the matrix polynomial* $\mathcal{T}(z)$ *are* $\mu_1 \geq \cdots \geq \mu_d$.
   (IV) *The matrix polynomial* $\mathcal{T}(z)$ *satisfies* (a) *through* (d) *in Theorem* 10.3.

Notice that the entries $\mu_1, \dots, \mu_d$ of $\mu$ in Theorem 10.4 (B) are the Forney indices of $\mathcal{T}(z)$, see the definition after the proof of Theorem 4.3.

**Proof of Theorem 10.4.** The claim in (A), with the exception of property (a) in Theorem 10.3 follows from Theorem 4.3. That in this case $\mathcal{P}(z)$ from Theorem 4.3 satisfies (a) in Theorem 10.3 follows from the assumption (i) and the fact that $\det \mathcal{W}(z) \not\equiv 0$.

The first part of (B) follows from [2, Theorem 1.5.7]; for this simple case we give a direct proof. The reproducing kernel space with kernel $K_{\mathcal{T}}$ is

$$\mathfrak{K}_{\mathcal{T}} = \operatorname{span}\big\{ K_{\mathcal{T}}(z,w)x \,:\, w \in \mathbb{C}, \ x \in \mathbb{C}^d \big\}.$$

Moreover, for every finite set $\mathbb{F} \subset \mathbb{C}$ we have

$$\mathfrak{K}_{\mathcal{T}} = \operatorname{span}\big\{ K_{\mathcal{T}}(z,w)x \,:\, w \in \mathbb{C} \setminus \mathbb{F}, \ x \in \mathbb{C}^d \big\}. \tag{10.9}$$

To prove (10.9) assume that $f(z) \in \mathfrak{K}_{\mathcal{T}}$ is orthogonal in $\mathfrak{K}_{\mathcal{T}}$ to the span on the right-hand side of the equality in (10.9). That is, assume that for every $w \in \mathbb{C} \setminus \mathbb{F}$ and every $x \in \mathbb{C}^d$ we have

$$0 = \big[f(z), K_{\mathcal{J}}(z,w)x\big]_{\mathfrak{K}_{\mathcal{J}}} = x^* f(w).$$

This yields that $f(w) = 0$ for all $w \in \mathbb{C} \setminus \mathbb{F}$, implying that $f = 0$ in $\mathfrak{K}_{\mathcal{J}}$.

Next choose $\mathbb{F}$ to be the finite set

$$\mathbb{F} = \big\{z \in \mathbb{C} : \det W(z) = 0\big\}.$$

In view of Theorem 10.3, the equality (10.9) for $\mathfrak{K}_{\mathcal{P}}$ reads

$$\mathfrak{C}_\mu = \mathfrak{K}_{\mathcal{P}} = \operatorname{span}\big\{K_{\mathcal{P}}(z,w)x \,:\, w \in \mathbb{C} \setminus \mathbb{F}, \ x \in \mathbb{C}^d\big\}. \tag{10.10}$$

Then, (10.9), (10.10), $K_{\mathcal{J}}(z,w) = W(z)K_{\mathcal{P}}(z,w)W(w)^*$ and the fact that $W(w)$ is invertible for all $w \in \mathbb{C} \setminus \mathbb{F}$, yield

$$\begin{aligned}
\mathfrak{K}_{\mathcal{J}} &= \operatorname{span}\big\{K_{\mathcal{J}}(z,w)y \,:\, w \in \mathbb{C} \setminus \mathbb{F}, \ y \in \mathbb{C}^d\big\} \\
&= \operatorname{span}\big\{W(z)K_{\mathcal{P}}(z,w)W(w)^*y \,:\, w \in \mathbb{C} \setminus \mathbb{F}, \ y \in \mathbb{C}^d\big\} \\
&= \operatorname{span}\big\{W(z)K_{\mathcal{P}}(z,w)x \,:\, w \in \mathbb{C} \setminus \mathbb{F}, \ x \in \mathbb{C}^d\big\} \\
&= \big\{W(z)f(z) \,:\, f(z) \in \mathfrak{C}_\mu\big\}.
\end{aligned}$$

What we just proved implies that $W$ is a surjection from $\mathfrak{K}_{\mathcal{P}} = \mathfrak{C}_\mu$ onto $\mathfrak{K}_{\mathcal{J}}$. That this operator is a linear injection is trivial. To verify that $W$ is an isometry we calculate with $f(z) \in \mathfrak{K}_{\mathcal{P}}$, $w \in \mathbb{C} \setminus \mathbb{F}$ and $x \in \mathbb{C}^d$:

$$\begin{aligned}
\big[W(z)f(z), W(z)K_{\mathcal{P}}(z,w)x\big]_{\mathfrak{K}_{\mathcal{J}}} &= \big[W(z)f(z), W(z)K_{\mathcal{P}}(z,w)W(w)^*W(w)^{-*}x\big]_{\mathfrak{K}_{\mathcal{J}}} \\
&= \big[W(z)f(z), K_{\mathcal{J}}(z,w)W(w)^{-*}x\big]_{\mathfrak{K}_{\mathcal{J}}} \\
&= \big(W(w)^{-*}x\big)^* W(w)f(w) \\
&= x^* f(w) \\
&= [f(z), K_{\mathcal{P}}(z,w)x]_{\mathfrak{K}_{\mathcal{P}}}.
\end{aligned}$$

This, (10.9) and (10.10) imply that $W$ is an isometry from $\mathfrak{K}_{\mathcal{P}} = \mathfrak{C}_\mu$ onto $\mathfrak{K}_{\mathcal{J}}$.

The equality $WS_{\mathfrak{C}_\mu} = S_{\mathfrak{K}_{\mathcal{J}}}W$ follows from the just proven equality $\mathfrak{K}_{\mathcal{J}} = W\mathfrak{C}_\mu$. That $S_{\mathfrak{K}_{\mathcal{J}}}$ is symmetric is a consequence of the facts that $S_{\mathfrak{C}_\mu}$ is symmetric, $W$ is an isometry and $S_{\mathfrak{K}_{\mathcal{J}}} = WS_{\mathfrak{C}_\mu}W^{-1}$.

The equivalence (I)⇔(II) in (C) follows from Theorem 6.2.

The implication (II)⇒(III) is easily verified. To prove the converse (III)⇒(II) assume (III). Since we assume that rank $P_\infty = d$, Theorem 4.1 implies that $\mathcal{P}(z)$ has the predictable degree property. This yields (10.8). Thus $W(z)$ has the block upper triangular form as in (6.1). Therefore $\det W(z)$ is constant. This constant is not zero since the assumption (ii) implies $\det W(z) \not\equiv 0$. Thus $W(z)$ is unimodular.

We have thus established that (I)⇔(II)⇔(III). Next we will prove that assuming any, and then all, of these three statements implies (IV). That $\mathcal{J}(z)$ satisfies (a) in

Theorem 10.3 is assumed in (i). That $\mathcal{T}(z)$ satisfies (b) in Theorem 10.3 follows from (II), and that it satisfies (d) in Theorem 10.3 follows from (III). That $\mathcal{T}(z)$ satisfies (c) in Theorem 10.3 follows from Corollary 4.2.

Finally, (IV)$\Rightarrow$(I) follows from Theorem 10.3.   $\square$

## 11. Finite-dimensional Pontryagin spaces

In Section 9 we have seen that each operator without eigenvalues in a finite-dimensional space is symmetrizable. In this section we prove that a symmetric operator without eigenvalues in a finite-dimensional Pontryagin space is isomorphic to the operator of multiplication by the independent variable in some canonical subspace of $\mathbb{C}^d[z]$ for some $d \in \mathbb{N}$ equipped with an inner product determined by a matrix polynomial as in Theorem 10.3. We give two proofs of the next theorem. In both proofs we apply results of this paper. In the first we invoke Theorem 10.4 and statements from Section 3, while in the second we use Theorem 7.1 and the "if" part of [10, Theorem 1.1].

**Theorem 11.1.** *Let $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ be a finite-dimensional Pontryagin space and let $S$ be a symmetric operator in $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ without eigenvalues. Let $\delta_S$ be the tuple of dimensions of $S$ and set $\mu = (\mu_1, \ldots, \mu_d) = \mathsf{Con}(\mathsf{Der}\,\delta_S)$ with $d = \mathrm{codim}(\mathrm{dom}\,S)$. Let $\mathsf{Q}$ be an arbitrary $2d \times 2d$ self-adjoint matrix with $d$ positive and $d$ negative eigenvalues. Then there exist:*

(A) *a matrix polynomial $\mathscr{P}(z) \in \mathbb{C}^{d \times 2d}[z]$ with properties (a) through (d) in Theorem 10.3 such that the reproducing kernel Pontryagin space with kernel $K_{\mathscr{P}}(z, w)$ defined in (10.1) is the canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$,*
(B) *an isomorphism $\Phi$ between the Pontryagin spaces $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ and $(\mathfrak{C}_\mu, [\,\cdot\,,\cdot\,]_{\mathfrak{C}_\mu})$ such that $\Phi S = S_{\mathfrak{C}_\mu} \Phi$.*

**Proof.** Let $S$ be as in the theorem. Let $S^*$ be the adjoint of $S$ in $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$.

**1.** Let $A$ be a self-adjoint operator extension of $S$ on $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ (see Lemma 3.5), let $p_A(z)$ be the characteristic polynomial of $A$ and let

$$R(z, A) := -p_A(z)(A - z)^{-1}, \quad z \in \rho(A),$$

be the scaled resolvent of $A$. Since $A$ is a self-adjoint operator on a Pontryagin space, the coefficients of $p_A(z)$ are real. For $u \in (\mathrm{dom}\,S)^{[\perp]}$, $v \in \mathrm{dom}\,S$ and $z \in \mathbb{C}$ we have

$$\begin{aligned}
\big[Sv, R(z, A)u\big]_{\mathfrak{G}} - \big[v, zR(z, A)u\big]_{\mathfrak{G}} &= \big[R(z^*, A)Sv, u\big]_{\mathfrak{G}} - \big[z^*R(z^*, A)v, u\big]_{\mathfrak{G}} \\
&= \big[R(z^*, A)(S - z^*)v, u\big]_{\mathfrak{G}} \\
&= -p_A(z^*)\big[v, u\big]_{\mathfrak{G}} \\
&= 0.
\end{aligned}$$

Hence, for arbitrary $z \in \rho(A)$ the mapping

$$u \mapsto \{R(z, A)u, zR(z, A)u\}, \quad u \in (\text{dom } S)^{[\perp]}, \tag{11.1}$$

is a linear injection defined on $(\text{dom } S)^{[\perp]}$ with values in $S^* \cap zI$. Since by Lemma 3.4

$$\dim\big((\text{dom } S)^{[\perp]}\big) = d \quad \text{and} \quad \dim\big(S^* \cap zI\big) = d \quad \text{for all} \quad z \in \mathbb{C},$$

the mapping in (11.1) is a bijection between $(\text{dom } S)^{[\perp]}$ and $S^* \cap zI$. Consequently,

$$\Big\{\{R(z, A)u, zR(z, A)u\} \, : \, u \in (\text{dom } S)^{[\perp]}\Big\} = S^* \cap zI \quad \text{for all} \quad z \in \rho(A). \tag{11.2}$$

Since $\text{dom}(S^* \cap zI) = \ker(S^* - zI)$, we also have

$$\ker(S^* - zI) = \Big\{R(z, A)u \, : \, u \in (\text{dom } S)^{[\perp]}\Big\} \quad \text{for all} \quad z \in \rho(A). \tag{11.3}$$

Set

$$T = \text{span}\Big\{\{R(z, A)u, zR(z, A)u\} : u \in (\text{dom } S)^{[\perp]}, \ z \in \rho(A)\Big\}. \tag{11.4}$$

Clearly $T \subseteq S^*$. Therefore $T \cap zI \subseteq S^* \cap zI$. Since for $z \in \rho(A)$ the subspace in (11.2) is a subset of $T \cap zI$, we conclude that

$$T \cap zI = S^* \cap zI \quad \text{for all} \quad z \in \rho(A). \tag{11.5}$$

Next we will prove that $T^*$ is an operator without eigenvalues. First notice that for all $z \in \rho(A)$ and all $u, v, x \in \mathfrak{G}$ we have

$$[v, R(z, A)x]_{\mathfrak{G}} - [u, zR(z, A)x]_{\mathfrak{G}} = \big[R(z^*, A)(v - z^*u), x\big]_{\mathfrak{G}}.$$

This identity, the definition of the adjoint and elementary considerations yield that the following four statements are equivalent:

(a)  $\{u, v\} \in T^*$.
(b)  For all $z \in \rho(A)$ and all $x \in (\text{dom } S)^{[\perp]}$ we have $\big[R(z^*, A)(v - z^*u), x\big]_{\mathfrak{G}} = 0$.
(c)  For all $z \in \rho(A)$ we have $(A - z^*)^{-1}(v - z^*u) \in \text{dom } S$.
(d)  For all $z \in \rho(A)$ we have $v - z^*u \in \text{ran}(S - z^*)$.

Therefore, if $\{0, v\} \in T^*$, then $v \in \text{ran}(S - z^*)$ for all $z \in \rho(A)$, which, by Corollary 3.3, implies that $v = 0$, proving that $T^*$ is an operator. If $\lambda \in \mathbb{C}$ and $\{u, \lambda u\} \in T^*$, then $(\lambda - z^*)u \in \text{ran}(S - z^*)$ for all $z \in \rho(A)$. Consequently, $u \in \text{ran}(S - z^*)$ for all $z \in \rho(A) \setminus \{\lambda\}$. Again, Corollary 3.3 yields $u = 0$, proving that $T^*$ does not have eigenvalues.

Since neither $T^*$ nor $S$ has eigenvalues, (11.5) and Lemma 3.4 imply that for every $z \in \mathbb{C}$ we have

$$
\begin{aligned}
\operatorname{codim}(\operatorname{dom} T^*) &= \operatorname{codim}\big((\operatorname{ran}(T^* - z))\big) \\
&= \dim\big(\operatorname{dom}(T \cap z^*I)\big) \\
&= \dim\big(\operatorname{dom}(S^* \cap z^*I)\big) \\
&= \operatorname{codim}(\operatorname{dom} S)
\end{aligned}
$$

Now $S \subseteq T^*$ yields $T^* = S$, and consequently $T = S^*$. An immediate consequence of $T = S^*$, (11.5) and (11.4) is

$$
\begin{aligned}
S^* &= \operatorname{span}\Big\{ \{R(z,A)u, zR(z,A)u\} : u \in (\operatorname{dom} S)^{[\perp]},\ z \in \rho(A) \Big\} \qquad (11.6) \\
&= \operatorname{span}\{ S^* \cap zI : z \in \rho(A) \}.
\end{aligned}
$$

**2.** Let $\mathsf{b} : S^* \to \mathbb{C}^{2d}$ be the boundary mapping for $S^*$ with Gram matrix $-\mathsf{Q}$. That is for every $\{x,y\}, \{u,v\} \in S^*$ we have

$$
-\mathsf{i}\big([y,u]_{\mathfrak{G}} - [x,v]_{\mathfrak{G}}\big) = \mathsf{b}(\{u,v\})^*(-\mathsf{Q})\mathsf{b}(\{x,y\}). \qquad (11.7)
$$

Let $\mathsf{b}_{\mathsf{ex}} : \mathfrak{G}^2 \to \mathbb{C}^{2d}$ be an arbitrary linear extension of $\mathsf{b} : S^* \to \mathbb{C}^{2d}$. Let

$$
u_1, \ldots, u_d
$$

be a basis for $(\operatorname{dom} S)^{[\perp]}$. Then for every $z \in \rho(A)$ the pairs

$$
\{R(z,A)u_j, zR(z,A)u_j\} \quad \text{with} \quad j \in \{1, \ldots, d\},
$$

form a basis for $S^* \cap zI$.

By Lemma 3.1 for every $z \in \rho(A)$ we have

$$
R(z,A) = \sum_{k=0}^{n-1} C_k z^k, \qquad (11.8)
$$

where $C_{n-1} = I$ and for $k \in \{0, \ldots, n-2\}$ the operator $C_k$ is a linear combination of powers of $A$, see (3.2). Notice that (11.8) implies that the operator $R(z,A)$ is defined for all $z \in \mathbb{C}$; however it is guaranteed to be a bijection whenever $z \in \rho(A)$.

Notice that

$$
zR(z,A) = (z - A + A)R(z,A) = p_A(z) + AR(z,A).
$$

Therefore, for each $j \in \{1, \ldots, d\}$ we have

$$\{R(z,A)u_j, zR(z,A)u_j\} = p_A(z)\{0,u_j\} + \{R(z,A)u_j, AR(z,A)u_j\}.$$

Further, by (11.8), for each $j \in \{1,\dots,d\}$ we have

$$\{R(z,A)u_j, zR(z,A)u_j\} = p_A(z)\{0,u_j\} + \sum_{k=0}^{n-1} z^k \{C_k u_j, AC_k u_j\}.$$

The linearity of $\mathsf{b}_{\mathsf{ex}}$ yields

$$\mathsf{b}\big(\{R(z,A)u_j, zR(z,A)u_j\}\big) = p_A(z)\mathsf{b}_{\mathsf{ex}}\big(\{0,u_j\}\big) + \sum_{k=0}^{n-1} z^k \mathsf{b}_{\mathsf{ex}}\big(\{C_k u_j, AC_k u_j\}\big).$$

Since the range of $\mathsf{b}_{\mathsf{ex}}$ is in $\mathbb{C}^{2d}$, the expression on the right-hand side is a vector polynomial in $\mathbb{C}^{2d}[z]$. Therefore,

$$\mathcal{B}(z) := \Big[\mathsf{b}\big(\{R(z,A)u_1, zR(z,A)u_1\}\big) \quad \cdots \quad \mathsf{b}\big(\{R(z,A)u_d, zR(z,A)u_d\}\big)\Big]$$

is a matrix polynomial in $\mathbb{C}^{2d\times d}[z]$. As $S\cap zI = \{\{0,0\}\}$, the mapping $\mathsf{b}$, when restricted to $S^* \cap zI$, is a bijection. Hence, for every $z \in \rho(A)$ we have rank $\mathcal{B}(z) = d$.

Notice that by (11.7) for arbitrary $j,k \in \{1,\dots,d\}$ and $z,w \in \rho(A)$ we have

$$\mathsf{b}\big(\{R(z^*,A)u_j, z^*R(z^*,A)u_j\}\big)^* \mathsf{Q}\mathsf{b}\big(\{R(w^*,A)u_k, w^*R(w^*,A)u_k\}\big)$$
$$= \mathrm{i}\Big(\big[w^*R(w^*,A)u_k, R(z^*,A)u_j\big]_{\mathfrak{G}} - \big[R(w^*,A)u_k, z^*R(z^*,A)u_j\big]_{\mathfrak{G}}\Big)$$
$$= -\mathrm{i}(z-w^*)\big[R(w^*,A)u_k, R(z^*,A)u_j\big]_{\mathfrak{G}}.$$

The last displayed expression is the entry in the $j$-th row and the $k$-th column of the $d \times d$ matrix polynomial $\mathcal{B}(z^*)^* \mathsf{Q}\mathcal{B}(w^*)$. Or, in a formula,

$$\mathcal{B}(z^*)^* \mathsf{Q}\mathcal{B}(w^*) = -\mathrm{i}(z-w^*)\Big[\big[R(w^*,A)u_k, R(z^*,A)u_j\big]_{\mathfrak{G}}\Big]_{j,k=1}^{d}, \quad z,w \in \rho(A).$$

Set

$$\mathcal{T}(z) = \big(\mathsf{Q}\mathcal{B}(z^*)\big)^*. \tag{11.9}$$

Then $\mathcal{T}(z) \in \mathbb{C}^{d\times 2d}[z]$ and for $z,w \in \rho(A)$

$$\mathcal{T}(z)\mathsf{Q}^{-1}\mathcal{T}(w)^* = \big(\mathsf{Q}\mathcal{B}(z^*)\big)^* \mathsf{Q}^{-1}\mathsf{Q}\mathcal{B}(w^*)$$
$$= \mathcal{B}(z^*)^* \mathsf{Q}\mathcal{B}(w^*)$$
$$= -\mathrm{i}(z-w^*)\Big[\big[R(w^*,A)u_k, R(z^*,A)u_j\big]_{\mathfrak{G}}\Big]_{j,k=1}^{d}. \tag{11.10}$$

Hence

$$\mathcal{T}(z)\mathsf{Q}^{-1}\mathcal{T}(z^*)^* = 0 \quad \text{for all} \quad z \in \mathbb{C}.$$

Further, since rank $\mathcal{B}(z) = d$ for all $z \in \rho(A)$, we have

$$\text{rank}\,\mathcal{T}(z) = d \quad \text{for all} \quad z \in \rho(A).$$

Set

$$K_{\mathcal{T}}(z, w) = \frac{\mathrm{i}}{z - w^*}\mathcal{T}(z)\mathsf{Q}^{-1}\mathcal{T}(w)^*, \quad w \ne z^*, \ z, w \in \mathbb{C}.$$

Then, by (11.10),

$$K_{\mathcal{T}}(z, w) = \left[\ \big[R(w^*, A)u_k, R(z^*, A)u_j\big]_{\mathfrak{G}}\ \right]_{j,k=1}^{d} \quad \text{for all} \quad z, w \in \rho(A). \qquad (11.11)$$

**3.** Let $\mathfrak{K}_{\mathcal{T}}$ be the reproducing kernel space with kernel $K_{\mathcal{T}}(z, w)$. It follows from Corollary 3.8 and (11.3) that

$$\mathfrak{G} = \text{span}\Big\{R(w^*, A)u_k\ :\ k \in \{1, \ldots, d\},\ w \in \rho(A)\Big\}. \qquad (11.12)$$

Consequently,

$$\mathfrak{K}_{\mathcal{T}} = \text{span}\big\{K_{\mathcal{T}}(z, w)x : x \in \mathbb{C}^d,\ w \in \mathbb{C}\big\}$$
$$= \left\{\left[\ [v, R(z^*, A)u_1]_{\mathfrak{G}}\ \cdots\ [v, R(z^*, A)u_d]_{\mathfrak{G}}\ \right]^{\top} : v \in \mathfrak{G}\right\}.$$

Now consider the mapping $\Psi : \mathfrak{G} \to \mathfrak{K}_{\mathcal{T}}$ defined by

$$(\Psi v)(z) := \left[\ [R(z, A)v, u_1]_{\mathfrak{G}}\ \cdots\ [R(z, A)v, u_d]_{\mathfrak{G}}\ \right]^{\top}, \quad z \in \rho(A), \quad v \in \mathfrak{G}.$$

The last expression for $\mathfrak{K}_{\mathcal{T}}$ implies that $\Psi$ is a surjection. To show that $\Psi$ is an injection assume $\Psi v = 0$. Then $(A - z)^{-1}v \in \text{dom}\,S$ for all $z \in \rho(A)$ and hence

$$v \in \bigcap\ \big\{\text{ran}(S - z)\ :\ z \in \rho(A)\big\}.$$

Consequently $v = 0$ by Corollary 3.8.

To prove that $\Psi$ is an isomorphism consider two special vectors from $\mathfrak{G}$:

$$v_1 = R(w_1^*, A)\sum_{k=1}^{d} x_k^1 u_k, \quad v_2 = R(w_2^*, A)\sum_{k=1}^{d} x_k^2 u_k,$$

where $w_1, w_2 \in \rho(A)$ and $x^1 = [x_1^1 \cdots x_d^1]^{\top}$, $x^2 = [x_1^2 \cdots x_d^2]^{\top} \in \mathbb{C}^d$. Clearly, using (11.11), we have

$$\big[v_1, v_2\big]_{\mathfrak{G}} = (x^2)^* K_{\mathcal{J}}(w_2, w_1) x^1.$$

By the definition of $\Psi$ we have

$$\Psi v_1 = K_{\mathcal{J}}(\cdot, w_1) x^1, \quad \Psi v_2 = K_{\mathcal{J}}(\cdot, w_2) x^2$$

and thus, by the reproducing kernel property of $\mathfrak{K}_{\mathcal{J}}(z,w)$ and (11.11),

$$\big[\Psi v_1, \Psi v_2\big]_{\mathfrak{K}_{\mathcal{J}}} = (x^2)^* K_{\mathcal{J}}(w_2, w_1) x^1.$$

Now (11.12) implies that $\Psi$ is an isomorphism.

**4.** Next we consider the relation $\Psi S^* \Psi^{-1} \subset (\mathfrak{K}_{\mathcal{J}})^2$. The definition of $\Psi$ and (11.11) yield

$$\Psi R(w^*, A) y = K_{\mathcal{J}}(\cdot, w) x \quad \text{and} \quad \Psi w^* R(w^*, A) y = w^* K_{\mathcal{J}}(\cdot, w) x,$$

where $w \in \mathbb{C}$, $x = [x_1 \cdots x_d]^\top \in \mathbb{C}^d$ and $y = x_1 u_1 + \cdots + x_d u_d$. The preceding displayed formulas and (11.6) lead to a formula for $\Psi S^* \Psi^{-1}$ which is analogous to (11.6):

$$\begin{aligned}
\Psi S^* \Psi^{-1} &= \big\{ \{\Psi u, \Psi v\} \ : \ \{u, v\} \in S^* \big\} \\
&= \operatorname{span}\Big\{ \{\Psi R(w^*, A) y, \Psi w^* R(w^*, A) y\} \ : \ y \in (\operatorname{dom} S)^{[\perp]},\ w \in \rho(A) \Big\} \\
&= \operatorname{span}\Big\{ \{K_{\mathcal{J}}(\cdot, w) x, w^* K_{\mathcal{J}}(\cdot, w) x\} \ : \ w \in \rho(A),\ x \in \mathbb{C}^d \Big\}.
\end{aligned}$$

The last expression for $\Psi S^* \Psi^{-1}$ makes calculating the adjoint of $\Psi S^* \Psi^{-1}$ in the reproducing kernel space $(\mathfrak{K}_{\mathcal{J}}, [\cdot, \cdot]_{\mathfrak{K}_{\mathcal{J}}})$ easy. For $\{f, g\} \in (\mathfrak{K}_{\mathcal{J}})^2$, $w \in \mathbb{C}$ and $x \in \mathbb{C}^d$ we have

$$\big[g, K_{\mathcal{J}}(\cdot, w) x\big]_{\mathfrak{K}_{\mathcal{J}}} - \big[f, w^* K_{\mathcal{J}}(\cdot, w) x\big]_{\mathfrak{K}_{\mathcal{J}}} = x^* \big(g(w) - w f(w)\big).$$

Therefore $\{f, g\} \in \big(\Psi S^* \Psi^{-1}\big)^*$ if and only if $g(z) = z f(z)$ and both $f(z), g(z) \in \mathfrak{K}_{\mathcal{J}}$. Thus the adjoint of $\Psi S^* \Psi^{-1}$ in the reproducing kernel space $\mathfrak{K}_{\mathcal{J}}$ is the operator $S_{\mathfrak{K}_{\mathcal{J}}}$ of multiplication by the independent variable in $\mathfrak{K}_{\mathcal{J}}$. Hence,

$$S_{\mathfrak{K}_{\mathcal{J}}} = \big(\Psi S^* \Psi^{-1}\big)^* = \Psi S \Psi^{-1}.$$

**5.** Finally, we apply statements (A) and (B) of Theorem 10.4 to $\mathcal{J}(z)$ to obtain the desired polynomial $\mathscr{P}(z)$. In the notation of Theorem 10.4 set $\Phi := W^{-1} \Psi$. Then $\Phi : \mathfrak{G} \to \mathfrak{C}_\mu$ is an isomorphism and

$$\Phi S = W^{-1} \Psi S = W^{-1} S_{\mathfrak{K}_{\mathcal{J}}} \Psi = S_{\mathfrak{C}_\mu} W^{-1} \Psi = S_{\mathfrak{C}_\mu} \Phi. \quad \square$$

As mentioned in the Introduction the research in this paper is closely connected to [10], where the main topic was the characterization of full matrix polynomial Nevanlinna kernels. By definition a kernel $K(z, w)$ is a full matrix polynomial Nevanlinna kernel if for some self-adjoint $2d \times 2d$ matrix $\mathsf{Q}$ with $d$ positive and $d$ negative eigenvalues there exists a $d \times 2d$ matrix polynomial $\mathscr{P}(z)$ such that

$$\mathscr{P}(z)\mathsf{Q}^{-1}\mathscr{P}(w)^* = -\mathrm{i}(z - w^*)K(z, w) \quad \text{for all} \quad z, w \in \mathbb{C} \tag{11.13}$$

and rank $\mathscr{P}(z) = d$ for all $z \in \mathbb{C}$. To show this connection more explicitly we give a second proof of Theorem 11.1.

**Second proof of Theorem 11.1.** Let $\Phi$ be the isomorphism between the vector space $\mathfrak{G}$ and the canonical space $\mathfrak{C}_\mu$ whose existence has been established in Theorem 7.1. Notice that $\Phi$ is first defined on a special basis of $\mathfrak{G}$ with values at the vectors $e_{d,k}z^j \in \mathfrak{C}_\mu$, $j \in \{0, \ldots, \mu_k - 1\}$, $k \in \{1, \ldots, d\}$, of the standard basis of $\mathfrak{C}_\mu$. Recall that $\mu_1 \geq \cdots \geq \mu_d$ and $\sum_{k=1}^d \mu_k = n = \dim \mathfrak{G}$.

Let $G$ be the Gram matrix of the vectors $\Phi^{-1}(e_{d,k}z^j) \in \mathfrak{G}$, $j \in \{0, \ldots, \mu_k - 1\}$, $k \in \{1, \ldots, d\}$, in this order. Denote by $\mathscr{B}(z)$ the $d \times n$ matrix polynomial whose columns are the vectors $e_{d,k}z^j \in \mathfrak{C}_\mu$, $j \in \{0, \ldots, \mu_k - 1\}$, $k \in \{1, \ldots, d\}$, in this order:

$$\mathscr{B}(z) = \begin{bmatrix} 1 & z & \cdots & z^{\mu_1-1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & z^{\mu_2-1} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & z^{\mu_d-1} \end{bmatrix}$$

Define the reproducing kernel

$$K_{\mathscr{B}}(z, w) := \mathscr{B}(z)G^{-1}\mathscr{B}(w)^*, \quad z, w \in \mathbb{C}.$$

We claim that the reproducing kernel space with kernel $K_{\mathscr{B}}(z, w)$ is exactly the canonical space $\mathfrak{C}_\mu$. Since $\mu_1 \geq \cdots \geq \mu_d$ and $\sum_{k=1}^d \mu_k = n$, we have $d\mu_1 \geq n$. With distinct $w_1, \ldots, w_{\mu_1} \in \mathbb{C}$ the $n \times d\mu_1$ matrix

$$\begin{bmatrix} \mathscr{B}(w_1)^* & \cdots & \mathscr{B}(w_{\mu_1})^* \end{bmatrix}$$

has full rank $n$. Therefore for each $\alpha, \beta \in \mathbb{C}^n$ there exist vectors $x_1, \ldots, x_{\mu_1}, y_1, \ldots, y_{\mu_1} \in \mathbb{C}^d$ such that

$$\alpha = G^{-1}\sum_{j=1}^{\mu_1} \mathscr{B}(w_j)^* x_j, \quad \beta = G^{-1}\sum_{j=1}^{\mu_1} \mathscr{B}(w_j)^* y_j.$$

For $f(z), g(z) \in \mathfrak{C}_\mu$ whose coordinates with respect to the standard basis are $\alpha$ and $\beta$ we have

$$f(z) = \sum_{j=1}^{\mu_1} K_{\mathcal{B}}(z, w_j) x_j = \mathcal{B}(z) \alpha, \quad g(z) = \sum_{j=1}^{\mu_1} K_{\mathcal{B}}(z, w_j) y_j = \mathcal{B}(z) \beta.$$

This proves that the reproducing kernel space with kernel $K_{\mathcal{B}}(z, w)$ is exactly the canonical space $\mathfrak{C}_\mu$.

Moreover, this construction shows that the inverse images $\Phi^{-1} f$ and $\Phi^{-1} g$ in $\mathfrak{G}$ are the vectors with the coordinates $\alpha$ and $\beta$, respectively. Now we can show that $\Phi$ is an isomorphism between the Pontryagin space $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_\mathfrak{G})$ and the reproducing kernel space $\mathfrak{C}_\mu$ with kernel $K_{\mathcal{B}}(z, w)$. We calculate:

$$[f, g]_{\mathfrak{C}_\mu} = \left[ \sum_{j=1}^{\mu_1} K_{\mathcal{B}}(z, w_i) x_i, \sum_{j=1}^{\mu_1} K_{\mathcal{B}}(z, w_j) y_j \right]_{\mathfrak{C}_\mu}$$

$$= \sum_{j=1}^{\mu_1} \sum_{j=1}^{\mu_1} y_j^* K_{\mathcal{B}}(w_j, w_i) x_i$$

$$= \left( \sum_{j=1}^{\mu_1} \mathcal{B}(w_j)^* y_j \right)^* G^{-1} \left( \sum_{j=1}^{\mu_1} \mathcal{B}(w_i)^* x_i \right)$$

$$= (G\beta)^* G^{-1} G\alpha$$

$$= \beta^* G\alpha$$

$$= \left[ \Phi^{-1} f, \Phi^{-1} g \right]_\mathfrak{G}.$$

Since $\Phi S_\mathfrak{G} = S_{\mathfrak{C}_\mu} \Phi$, the operator $S_{\mathfrak{C}_\mu}$ is symmetric in the reproducing kernel space $\mathfrak{C}_\mu$ with kernel $K_{\mathcal{B}}(z, w)$. Moreover, for all $\alpha \in \mathbb{C}$ we have

$$\operatorname{ran}(S_{\mathfrak{C}_\mu} - \alpha) = \{ f \in \mathfrak{C}_\mu : f(\alpha) = 0 \}.$$

By the "if" part of [10, Theorem 1.1] the reproducing kernel $K_{\mathcal{B}}(z, w)$ is a full matrix polynomial Nevanlinna kernel. The equality (11.13) implies that the row degrees of $\mathcal{P}(z)$ are given by $\mu_1, \ldots, \mu_d$. $\quad \square$

## 12. Examples

**Example 12.1.** Consider the following operator in $\mathbb{C}^6$

$$S = \left\{ \{x, y\} : x, y \in \mathbb{C}^6, \begin{array}{c} y_2 = y_3 = x_1, \ y_4 = x_3 = x_2, \\ y_5 = y_1 = x_6 = x_4 = 0, \ y_6 = x_5 \end{array} \right\}$$

$$= \operatorname{span}\{\{e_{6,1}, e_{6,2} + e_{6,3}\}, \{e_{6,2} + e_{6,3}, e_{6,4}\}, \{e_{6,5}, e_{6,6}\}\}.$$

Clearly,

$$S^2 = \operatorname{span}\{\{e_{6,1}, e_{6,4}\}\}$$

and hence,

$$\dim S^0 = 6, \quad \dim S = 3, \quad \dim S^2 = 1, \quad \dim S^3 = 0.$$

Consequently $S$ has no eigenvalues and $\delta = (6,3,1)$, $\nu = \mathsf{Der}\,\delta = (3,2,1)$ and $\mu = \mathsf{Con}\,\nu = (3,2,1)$. The Young diagram associated with this operator is ⊞.

By Theorem 7.1 $S$ is similar to the operator of multiplication with the independent variable in the canonical space

$$\mathfrak{C}_{(3,2,1)} = e_{3,1}\mathbb{C}[z]_{<3} \dotplus e_{3,2}\mathbb{C}[z]_{<2} \dotplus e_{3,3}\mathbb{C}[z]_{<1}.$$

The linear bijection $\Phi : \mathbb{C}^6 \to \mathfrak{C}_{(3,2,1)}$ from Theorem 7.1 (II) which intertwines $S$ and $S_{\mathfrak{C}_{(3,2,1)}}$ is given by:

$$e_{6,1} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_{6,2} + e_{6,3} \mapsto \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \quad e_{6,4} \mapsto \begin{bmatrix} z^2 \\ 0 \\ 0 \end{bmatrix},$$

$$e_{6,5} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad\quad e_{6,6} \mapsto \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix},$$

$$e_{6,2} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For this $S$, the nilpotent operator $N$ from Theorem 8.3 is given by the following matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and the nilpotent operator $D$ from Theorem 8.4 is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the direct sum decomposition of $\mathbb{C}^6$ from Corollary 9.2 we have

$$\mathfrak{F}_0 = \mathrm{span}\{e_{6,2}\}, \quad \mathfrak{F}_1 = \mathrm{span}\{e_{6,5}, e_{6,6}\}, \quad \mathfrak{F}_2 = \mathrm{span}\{e_{6,1}, e_{6,2}+e_{6,3}, e_{6,4}\}.$$

The inner products studied in the proof of Theorem 9.3 in which $S$ is symmetric are given by the Gram matrices

$$
G = \begin{bmatrix}
h_5 & 0 & h_6 & h_7 & 0 & 0 \\
0 & h_1 & -h_1 & 0 & 0 & 0 \\
h_6 & -h_1 & h_1 + h_7 & h_8 & 0 & 0 \\
h_7 & 0 & h_8 & h_9 & 0 & 0 \\
0 & 0 & 0 & 0 & h_2 & h_3 \\
0 & 0 & 0 & 0 & h_3 & h_4
\end{bmatrix}
$$

relative to the standard basis of $\mathbb{C}^6$. Here $h_1, \ldots, h_9$ are real and such that the matrices are invertible. The nilpotent matrix $N$ is self-adjoint in these inner products, that is $GN = N^*G$, if and only if $h_4 = h_5 = h_9 = 0$. In this case the above Gram matrices are invertible if and only if $h_3 h_7 h_9 \neq 0$. It follows from Theorem 5.1.1 and Corollary 5.1.2 in [16] that such inner products cannot be positive definite. The matrix $G$ is positive definite if for example

$$
h_5 = h_9 = h_2 = h_4 = 2, \quad h_6 = h_8 = -1, \quad h_7 = h_3 = h_1 = 1.
$$

**Example 12.2.** Consider the vector space $\mathbb{C}^6$ with the indefinite inner product $[x, y] = \langle Jx, y \rangle$ where

$$
J = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

is a self-adjoint involution matrix and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{C}^6$. Let

$$
S = \left\{ \{x, y\} \, : \, x, y \in \mathbb{C}^6, \quad \begin{matrix} y_6 = y_4 = x_5 = x_2, & y_5 = 0, & y_2 = x_1, \\ x_3 = 0, & -y_3 = y_1 = x_6 = x_4, \end{matrix} \right\}
$$

$$
= \operatorname{span}\left\{ \{e_{6,1}, e_{6,2}\}, \{e_{6,2} + e_{6,5}, e_{6,4} + e_{6,6}\}, \{e_{6,4} + e_{6,6}, e_{6,1} - e_{6,3}\} \right\}
$$

be a symmetric operator in $(\mathbb{C}^6, [\cdot, \cdot])$. Its adjoint is

$$
S^{[*]} = \left\{ \{x, y\} \, : \, x, y \in \mathbb{C}^6, \quad \begin{matrix} y_3 = -x_4 + x_6 - y_1, \\ y_5 = x_1 + x_3 - y_2, & y_6 = x_2 \end{matrix} \right\}.
$$

A boundary mapping $\mathsf{b} : S^{[*]} \to \mathbb{C}^6$ is given by

$$\mathsf{b}\big(\{x,y\}\big) = \begin{bmatrix} x_1 - y_2 & x_2 - y_4 & x_3 & x_4 - y_1 & x_5 - y_4 & x_6 - y_1 \end{bmatrix}^\top,$$

where $\{x,y\} \in S^{[*]}$ and the corresponding Gram matrix is

$$Q = \mathrm{i} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

The self-adjoint extension of $JS$ in $(\mathbb{C}^6, \langle\,\cdot\,,\cdot\,\rangle)$ introduced in Lemma 3.5 leads to the following self-adjoint extension of $S$ in $(\mathbb{C}^6, [\,\cdot\,,\cdot\,])$:

$$A = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}J \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The scaled resolvent (see the beginning of Section 3) of $A$ is

$$R(z, A) = -4 \det(z - A)(A - z)^{-1}$$

$$= \begin{bmatrix} 4z^5 + 2z^2 & 4z^3 + 1 & 2z^2 & -2z & -1 & 4z^4 + 2z \\ 4z^4 + z & 4z^5 + z^2 & 2z^4 & -2z^3 & -z^2 & 4z^3 + 1 \\ -2z^2 & -2z^3 & 4z^5 - 4z^2 & 4z - 4z^4 & 2 - 2z^3 & -2z \\ 2z^3 & 2z^4 & 2z^3 - 1 & 4z^5 - 4z^2 & 2z^4 - 2z & 2z^2 \\ -z & -z^2 & 2z^4 - 2z & 2 - 2z^3 & 4z^5 - 3z^2 & -1 \\ 4z^3 + 1 & 4z^4 + z & 2z^3 & -2z^2 & -z & 4z^5 + 2z^2 \end{bmatrix}.$$

Here we additionally scale the resolvent by 4 to avoid fractions. Choose a basis of $(\operatorname{dom} S)^{[\perp]}$ to be

$$\big\{ [0, -1, 0, 0, 1, 0]^\top, \; [0, 0, 0, 1, 0, 0]^\top, [1, 0, -1, 0, 0, 0]^\top \big\}.$$

Then the matrix polynomial $\mathcal{T}(z)$ from (11.9) in the proof of Theorem 11.1 is given by

$$\mathcal{T}(z) = \mathrm{i} \begin{bmatrix} 8z^5 & 4z^6 - 2z^3 - 2 & 4z^5 & 0 & 2z^3 - 4z^6 & 4z^4 \\ 2 & 2z^4 - 2z & -4z^6 + 2z^3 + 2 & 0 & 2z^4 - 2z & 4z^5 - 2z^2 \\ -4z^4 & 2z^5 - z^2 & -2z^4 & 4z^6 - 2z^3 - 1 & 2z^5 - z^2 & -4z^6 + 1 \end{bmatrix}.$$

To get the polynomial $\mathscr{P}(z)$ whose existence is claimed in Theorem 11.1 we use the Smith normal form of $\mathcal{T}(z)$ to obtain the factorization

$$\mathcal{T}(z) = i \begin{bmatrix} 8z^5 & -8z^9 + 12z^6 - 2z^3 - 2 & -8z^7 + 4z^4 + 2z \\ 2 & 0 & 0 \\ -4z^4 & 4z^8 - 2z^5 - z^2 & 4z^6 - 2z^3 - 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & z^4 - z & -2z^6 + z^3 + 1 & 0 & z^4 - z & 2z^5 - z^2 \\ 0 & 1 & 0 & z & 0 & -z \\ 0 & 0 & -2z^4 & 1 - z^3 & z^2 & 3z^3 - 1 \end{bmatrix}.$$

Further we apply the method of the proof of Theorem 4.3 to get the factorization the last $3 \times 6$ matrix as follows

$$\begin{bmatrix} -2z^3 + 1 & z^2 & -z \\ 0 & 0 & 1 \\ -2z & 1 & -z^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & z^3 + 1 & 0 & -z & -z^2 \\ 2z & z^2 & 2z & 1 & -z^2 & -1 \\ 0 & 1 & 0 & z & 0 & -z \end{bmatrix}.$$

Hence

$$\mathcal{T}(z) = 2i \begin{bmatrix} -2z^2 & 2z^4 + z & -2z^3 - 1 \\ -2z^3 + 1 & z^2 & -z \\ z & -z^3 - 1/2 & 2z^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & z^3 + 1 & 0 & -z & -z^2 \\ 2z & z^2 & 2z & 1 & -z^2 & -1 \\ 0 & 1 & 0 & z & 0 & -z \end{bmatrix}.$$

This yields the desired polynomial $\mathscr{P}(z)$:

$$\mathscr{P}(z) = \begin{bmatrix} 1 & 0 & z^3 + 1 & 0 & -z & -z^2 \\ 2z & z^2 & 2z & 1 & -z^2 & -1 \\ 0 & 1 & 0 & z & 0 & -z \end{bmatrix}.$$

Notice that $\mathscr{P}(z)$ has rank 3 for all $z \in \mathbb{C}$ and that

$$P_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

also has rank 3. Hence, $3, 2, 1$ are the Forney indices of the matrix polynomials $\mathscr{P}(z)$ and $\mathcal{T}(z)$. Further, by Theorem 11.1 and the above construction we have that $\mathscr{P}(z)\mathsf{Q}^{-1}\mathscr{P}(z^*)^*$ $= 0$ for all $z \in \mathbb{C}$ and the row degrees of $\mathscr{P}(z)$ satisfy $3 > 2 > 1$. Thus, $\mathscr{P}(z)$ satisfies (a) through (d) in Theorem 10.3.

The corresponding reproducing kernel is

$$K(z, w) = \frac{i}{z - w^*} \mathscr{P}(z)\mathsf{Q}^{-1}\mathscr{P}(w)^* = \begin{bmatrix} 1 - w^*z^2 - w^{*2}z & -z + w^* & z^2 \\ z - w^* & 2zw^* & 0 \\ w^{*2} & 0 & 0 \end{bmatrix}.$$

The isomorphism $\Phi$ from Theorem 11.1 between the Pontryagin spaces $(\mathbb{C}^6, [\,\cdot\,,\cdot\,])$ and the reproducing kernel space $\mathfrak{C}_{(3,2,1)}$ with the reproducing kernel $K(z,w)$ such that $\Phi S = S_{\mathfrak{C}_\mu} \Phi$ is given by

$$e_{6,2} + e_{6,5} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_{6,4} + e_{6,6} \mapsto \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \quad e_{6,1} - e_{6,3} \mapsto \begin{bmatrix} z^2 \\ 0 \\ 0 \end{bmatrix},$$

$$-e_{6,1} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad -e_{6,2} \mapsto \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix},$$

$$e_{6,6} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Or, equivalently $\Phi$ is given by,

$$
\begin{aligned}
e_{6,1} &\mapsto -e_{3,2} & &= K(z,0)e_{3,3} + \tfrac{1}{2}K(z,1)e_{3,1} - \tfrac{1}{2}K(z,-1)e_{3,1}, \\
e_{6,2} &\mapsto -ze_{3,2} & &= K(z,0)(e_{3,1} + e_{3,2}) - K(z,1)e_{3,2}, \\
e_{6,3} &\mapsto -z^2 e_{3,1} - e_{3,2} &= \tfrac{1}{2}K(z,1)e_{3,1} - \tfrac{1}{2}K(z,-1)e_{3,1}, \\
e_{6,4} &\mapsto ze_{3,1} - e_{3,3} & &= K(z,0)e_{3,1} - \tfrac{1}{2}K(z,1)e_{3,1} - \tfrac{1}{2}K(z,-1)e_{3,1}, \\
e_{6,5} &\mapsto e_{3,1} + ze_{3,2} & &= K(z,0)e_{3,1}, \\
e_{6,6} &\mapsto e_{3,3} & &= -K(z,0)(e_{3,1} + e_{3,2}) + \tfrac{1}{2}K(z,1)e_{3,1} + \tfrac{1}{2}K(z,-1)e_{3,1}.
\end{aligned}
$$

The matrix $J$ is the Gram matrix of the vectors on the left-hand side in the space $(\mathbf{C}^6, [\,\cdot\,,\cdot\,])$. A lengthy but straightforward computation of the Gram matrix of the vectors on the right-hand side in the reproducing kernel Pontryagin space $\mathfrak{C}_{(3,2,1)}$ shows that this matrix also equals $J$. This confirms that $\Phi$ is an isomorphism between Pontryagin spaces.

For calculations in the above examples we used a Wolfram Mathematica package developed for explorations of operators without eigenvalues and matrix polynomials. This package is available on the first author's website.

**Declaration of competing interest**

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