CHARACTERISTIC FUNCTIONS OF UNITARY COLLIGATIONS AND OF BOUNDED OPERATORS IN KREIN SPACES

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Dedicated to Israel Gohberg
on the occasion of his sixtieth birthday

In this note we give necessary and sufficient conditions for a holomorphic operator valued function to coincide weakly with the characteristic function of a bounded operator on a Krein space. We also present a sufficient condition such that the weak isomorphism is an isomorphism, i.e., is bounded.

1. Introduction

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Krein spaces. By $\mathbf{S}(\mathfrak{F},\mathfrak{G})$ we denote the (generalized Schur) class of all functions Θ, defined and holomorphic at $z=0$ and with values in $\mathbf{L}(\mathfrak{F},\mathfrak{G})$, the space of bounded linear operators from $\mathfrak{F}$ to $\mathfrak{G}$ (we write $\mathbf{L}(\mathfrak{F})$ for $\mathbf{L}(\mathfrak{F},\mathfrak{F})$). It is known, see Section 3 below, that any $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ is in some neighborhood of $z=0$ the characteristic function Θ_Δ of a unitary colligation $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};U)$ with outer spaces $\mathfrak{F},\mathfrak{G}$ and a Krein space $\mathfrak{K}$ as inner space, that is, there exists a unitary mapping U with matrix representation

$$(1.1)\qquad U=\begin{pmatrix} T & F \\ G & H\end{pmatrix}:\begin{pmatrix}\mathfrak{K}\\ \mathfrak{F}\end{pmatrix}\to\begin{pmatrix}\mathfrak{K}\\ \mathfrak{G}\end{pmatrix}$$

such that the relation

$$(1.2)\qquad \Theta(z)=\Theta_\Delta(z)=H+zG(I-zT)^{-1}F$$

holds for all z in this neighborhood of 0. Here, e.g., $\left(\begin{smallmatrix}\mathfrak{K}\\ \mathfrak{F}\end{smallmatrix}\right)$ stands for the Krein space which is the orthogonal sum $\mathfrak{K}\oplus\mathfrak{F}$ of $\mathfrak{K}$ and $\mathfrak{F}$ (for details see Section 3).

Now let $\mathfrak{K}$ be an arbitrary Krein space and $T\in\mathbf{L}(\mathfrak{K})$. We fix a fundamental symmetry $J_{\mathfrak{K}}$ on $\mathfrak{K}$ and denote by T^* the adjoint of T with respect to the

Hilbert inner product on $\mathfrak{K}$ generated by $J_{\mathfrak{K}}$. We define the operators

$$(1.3) \quad J_T = \operatorname{sgn}(J_{\mathfrak{K}} - T^* J_{\mathfrak{K}} T), \; D_T = |J_{\mathfrak{K}} - T^* J_{\mathfrak{K}} T|^{1/2}$$

and set $\mathfrak{D}_T = \mathfrak{R}(D_T)^c$, the closure of the range $\mathfrak{R}(D_T)$. Then with T there is associated the unitary colligation $\Delta_T = (\mathfrak{K}, \mathfrak{D}_{T^*}, \mathfrak{D}_T; U_T)$ with

$$(1.4) \quad U_T = \begin{pmatrix} T & D_{T^*}|_{\mathfrak{D}_{T^*}} \\ D_T & -J_T L_{T^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_{T^*} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_T \end{pmatrix}.$$

Here L_{T^*} is the so-called link operator, introduced by Gr. Arsene, T. Constantinescu and A. Gheondea in [ACG] (for details see Section 4). The characteristic function of the colligation Δ_T:

$$(1.5) \quad \Theta_T(z) = -J_T L_{T^*} + z D_T (1 - zT)^{-1} D_{T^*},$$

defined for all z in a neighborhood of $z = 0$, is also called the characteristic function of the operator T (relative to $J_{\mathfrak{K}}$). It is the main purpose of this paper, to prove in Theorem 5.2 necessary and sufficient conditions for a given function $\Theta \in \mathbf{S}(\mathfrak{F}, \mathfrak{G})$ to coincide in a neighborhood of $z = 0$ with the characteristic function of some $T \in \mathbf{L}(\mathfrak{K})$. In other words, these conditions are equivalent to the fact, that the operators F, G, H and the spaces $\mathfrak{F}$, $\mathfrak{G}$ in (1.1) or (1.2) can be "identified" with the operators $D_{T^*}|_{\mathfrak{D}_{T^*}}$, D_T, $-J_T L_{T^*}$ and the spaces $\mathfrak{D}_{T^*}$, $\mathfrak{D}_T$, respectively, in (1.4) or (1.5), which are all determined by the operator T in the left upper corner of the matrix in (1.4) representing U. If, in particular, the Krein space $\mathfrak{K}$ is a Hilbert space, then $J_{\mathfrak{K}} = I$, $L_T = T$ and the characteristic function in (1.5) becomes the well-known function

$$\Theta_T(z) = -T^* J_{T^*} + z D_T (1 - zT)^{-1} D_{T^*}.$$

In this case such necessary and sufficient conditions for the equality $\Theta = \Theta_T$ to hold were proved by J.A. Ball, see [Ba], and we show in Section 6 how his result (formulated below as Theorem 6.1) follows from Theorem 5.2.

In these representation theorems for functions $\Theta \in \mathbf{S}(\mathfrak{F}, \mathfrak{G})$ the question arises of the uniqueness of the corresponding colligation Δ or the operator T. As is typical for Krein spaces, the colligation Δ or the Krein space $\mathfrak{K}$ and the operators in it are, in general, only unique up to weak isomorphisms. Recall that two Krein spaces $\mathfrak{K}$, $\mathfrak{K}'$ are called weakly isomorphic, if there is a (closed) linear mapping V from a dense subspace $\mathfrak{D} \subset \mathfrak{K}$ onto a dense subspace of $\mathfrak{K}'$ such that $[Vx,Vy]=[x,y]$, $x,y\in\mathfrak{D}$. In Section 7 (Theorem 7.1) we give a sufficient condition, which assures that such a weak isomorphism V is an isomorphism, i.e., that V is continuous.

At several places in this paper we use an extension of a basic result of M.G. Krein about operators in "spaces with two norms", which is formulated in Section 2. To conclude the outline of the contents of this paper we mention, that we do not repeat the (lengthy) proof of the basic statement (Theorem 3.4), that any $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ is the characteristic function of some unitary colligation Δ, see, e.g., [DLS2]. However, in Proposition 3.6 we prove this result in the particular case of Θ being holomorphic and unitary on an arc of the boundary $\partial\mathbb{D}$ of the open unit disc $\mathbb{D}$ in $\mathbb{C}$. Then the proof in [DLS2] can be simplified considerably.

In the sequel we shall use, as we did above, without further specification the elementary facts and the common notations from the theory of linear operators in Krein and Pontryagin spaces. They can be found in [AI], [Bo], [Cu], [DLS1], [DLS2], [IKL] and [L]. Here we only recall the notation that, if T is a densely defined operator from $\mathfrak{F}$ to $\mathfrak{G}$, T^+ denotes the Krein space adjoint of T: $T^+=J_{\mathfrak{F}}T^*J_{\mathfrak{G}}$, where $J_{\mathfrak{F}}$ and $J_{\mathfrak{G}}$ are fundamental symmetries on $\mathfrak{F}$ and $\mathfrak{G}$, respectively and T^* denotes the adjoint of T with respect to the Hilbert space structures on $\mathfrak{F}$ and $\mathfrak{G}$ induced by these symmetries. We shall use $[.,.]$ as the generic notation for the possibly indefinite inner product and $(.,.)$ in the cases, where we want to emphasize that the inner product is positive definite.

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2. Extension of a result of M.G. Krein on operators in spaces with two norms

Let $(\mathfrak{H},(.,.))$ be a Hilbert space and let $[.,.]$ be a (possibly indefinite) inner product (hermitian, sesquilinear form) on $\mathfrak{H}$ which is bounded, i.e., there exists a constant $C>0$ such that $|[x,y]|\leq C(x,x)^{1/2}(y,y)^{1/2}$, $x,y\in\mathfrak{H}$. By the Riesz' representation theorem there exists a selfadjoint $G\in\mathbf{L}(\mathfrak{H})$ such that $[x,y]=(Gx,y)$, $x,y\in\mathfrak{H}$. G is called the Gram operator. We may write the Hilbert space $\mathfrak{H}$ as the orthogonal sum $\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-\oplus\mathfrak{H}_0$, where

$\mathfrak{H}_\pm = \nu(I \mp \operatorname{sgn} G)$ and $\mathfrak{H}_0 = \nu(G)$. Here the sign function is defined by $\operatorname{sgn}(t) = -1$, 0 or 1, if $t<0$, $t=0$ or $t>0$, respectively, and $\nu(T)$ stands for the null space of the operator T. The spaces $\mathfrak{H}_\pm$ equipped with the inner product $\pm[x,y]$ are pre-Hilbert spaces. We denote their completions by $\mathfrak{K}_\pm$ and retain the notation $\pm[x,y]$ for their inner products. We put $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$, direct sum, and denote by $P_\pm$ the projections of $\mathfrak{K}$ onto $\mathfrak{K}_\pm$ along $\mathfrak{K}_\mp$. Then $(x,y) = [P_+x, P_+y] - [P_-x, P_-y]$ turns $\mathfrak{K}$ into a Hilbert space and $[x,y] = [P_+x, P_+y] + [P_-x, P_-y]$ turns $\mathfrak{K}$ into a Krein space with fundamental symmetry $J_{\mathfrak{K}} = P_+ - P_-$. We call $\mathfrak{K}$ the Krein space associated with the Hilbert space $\mathfrak{H}$ and the bounded inner product $[.,.]$. Clearly, $\mathfrak{H}_0 = \{x \in \mathfrak{H} \mid [x,y] = 0 \text{ for all } y \in \mathfrak{H}\}$ and the factor space $\hat{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H}_0$, consisting of equivalence classes $\hat{x} = \{x+y \mid y \in \mathfrak{H}_0\}$, can be identified with $\mathfrak{H}_+ \oplus \mathfrak{H}_-$ and is dense in $\mathfrak{K}$.

The following theorem is a generalization of a result announced by M.G. Krein in 1937 and published in [Kr] in 1947 (see also [AI] p. 220 and [Bo] p. 92, p. 98), which was later also proved in a similar form independently by W.T. Reid [Re] (see also [Ha], p.51), P.D. Lax [La] and J. Dieudonné [Di]. For the proof of the theorem, as Krein's proof based on repeated application of the Cauchy–Schwarz inequality, we refer to [DLS2].

THEOREM 2.1. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and suppose that on each $\mathfrak{H}_j$ there is given a bounded, and in general indefinite, inner product $[.,.]_j$, $j=1,2$. Furthermore, assume that we are given two operators $U_0 \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $V_0 \in \mathbf{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ such that $[U_0x, y]_2 = [x, V_0y]_1$, $x \in \mathfrak{H}_1$, $y \in \mathfrak{H}_2$. Then the operators $\hat{U}_0 \in \mathbf{L}(\hat{\mathfrak{H}}_1, \hat{\mathfrak{H}}_2)$ and $\hat{V}_0 \in \mathbf{L}(\hat{\mathfrak{H}}_2, \hat{\mathfrak{H}}_1)$ obtained from U_0 and V_0 on the factor spaces $\hat{\mathfrak{H}}_1$ and $\hat{\mathfrak{H}}_2$ in the usual way, can be extended by continuity to operators $U \in \mathbf{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ and $V \in \mathbf{L}(\mathfrak{K}_2, \mathfrak{K}_1)$, respectively, between the associated Krein spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$ and $[Ux, y]_2 = [x, Vy]_1$, $x \in \mathfrak{K}_1$, $y \in \mathfrak{K}_2$, i.e., $V = U^+$.*

For later reference we amplify two special cases of Theorem 2.1:

COROLLARY 2.2. *Let $\mathfrak{H}$ be a Hilbert space, let $[.,.]$ be a bounded inner product on $\mathfrak{H}$ and let $S_0 \in \mathbf{L}(\mathfrak{H})$ be such that $[S_0x, y] = [x, S_0y]$, $x, y \in \mathfrak{H}$. Then $\hat{S}_0 \in \mathbf{L}(\hat{\mathfrak{H}}_0)$ can be extended by continuity to a bounded selfadjoint operator $S = S^+$ on the associated Krein space $\mathfrak{K}$. If, moreover, $[.,.]$ is positive definite on $\mathfrak{H}$, then $\mathfrak{K}$ is a Hilbert space containing $\mathfrak{H}$ as a dense subspace and S_0 itself can be extended by continuity to a bounded selfadjoint operator $S = S^*$ on $\mathfrak{K}$.*

COROLLARY 2.3. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and suppose that on* *each $\mathfrak{H}_j$ there is given a bounded inner product* $[.,.]_j$, $j=1,2$. *Furthermore, assume that we are given a bijective operator* $U_0 \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ *such that* $[U_0 x, U_0 y]_2 = [x,y]_1$, $x,y \in \mathfrak{H}_1$. *Then* $\hat{U}_0 \in \mathbf{L}(\hat{\mathfrak{H}}_1, \hat{\mathfrak{H}}_2)$ *can be extended by continuity to a unitary operator* $U \in \mathbf{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ *between the associated Krein spaces* $\mathfrak{K}_1$ *and* $\mathfrak{K}_2$.

Proof. Apply Theorem 2.1 with $V_0 = U_0^{-1}$ and let U and V be the continuous extensions of $\hat{U}_0$ and $\hat{V}_0$ with $V = U^+$. From $[\hat{U}_0 x, \hat{U}_0 y]_2 = [x,y]_1$, $x,y \in \hat{\mathfrak{H}}_1$ it follows by continuity that $[Ux, Uy]_2 = [x,y]_1$, $x,y \in \mathfrak{K}_1$, that is $U^+U = I_{\mathfrak{K}_1}$. Similarly, one can show that $V^+V = I_{\mathfrak{K}_2}$ and therefore $UU^+ = I_{\mathfrak{K}_2}$. Hence U is unitary.

For more results in the theory of operators in spaces with two norms we refer to [GZ1], [GZ2].

3. Unitary colligations and their characteristic functions, the general representation theorem

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Krein spaces. If $\Theta \in \mathbf{S}(\mathfrak{F}, \mathfrak{G})$ we denote by $\mathfrak{D}(\Theta)$ the domain of holomorphy of Θ in $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and by $\mathsf{S}_\Theta^m(z,w)$ the matrix kernel

$$\mathsf{S}_\Theta^m(z,w) = \begin{pmatrix} \dfrac{I-\Theta(w)^+\Theta(z)}{1-\bar{w}z} & \dfrac{\Theta(\bar{z})^+-\Theta(w)^+}{z-\bar{w}} \\ \dfrac{\Theta(\bar{w})-\Theta(z)}{\bar{w}-z} & \dfrac{I-\Theta(\bar{w})\Theta(\bar{z})^+}{1-\bar{w}z} \end{pmatrix}$$

defined for z and w in a neighborhood of 0 contained in $\mathbb{D}$, $z \neq \bar{w}$ with values in $\mathbf{L}\begin{pmatrix}\mathfrak{F}\\ \mathfrak{G}\end{pmatrix}$. Clearly, $\mathsf{S}_\Theta^m(z,w)^+ = \mathsf{S}_\Theta^m(z,w)$. A kernel $\mathsf{K}(z,w)$ defined for z,w in some set $\mathfrak{D} \subset \mathbb{C}$ and with values in $\mathbf{L}(\mathfrak{K})$, where $\mathfrak{K}$ is a Krein space, is called positive definite, positive semidefinite, etc., if $\mathsf{K}(z,w)^+ = \mathsf{K}(w,z)$, so that all matrices of the form $([\mathsf{K}(z_i,z_j)f_i,f_j])_{i,j=1}^n$, where $n \in \mathbb{N}$, $z_1,\dots,z_n \in \mathfrak{D}$ and $f_1,\dots,f_n \in \mathfrak{K}$ are arbitrary, are hermitian, and the eigenvalues of each of these matrices are positive, nonnegative, etc., respectively. More generally, we say that $\mathsf{K}(z,w)$ has κ positive (negative) squares if $\mathsf{K}(z,w)^+ = \mathsf{K}(w,z)$ and all these hermitian matrices have at most κ and at least one has exactly κ positive (negative) eigenvalues. It has infinitely many positive (negative) squares if for each κ at least one of these matrices has not less than κ positive (negative) eigenvalues. A Pontryagin space of (negative) index κ, for example, is a Krein space on which the constant kernel $\mathsf{K}(z,w) = I$ has κ negative squares.

Finally, $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ and $\Theta'\in\mathbf{S}(\mathfrak{F}',\mathfrak{G}')$ are said to coincide, if there are two unitary operators $V\in\mathbf{L}(\mathfrak{F}',\mathfrak{F})$ and $W\in\mathbf{L}(\mathfrak{G}',\mathfrak{G})$ such that $\Theta(z)V=W\Theta'(z)$ for z in some neighborhood of 0 contained in the intersection $\mathfrak{D}(\Theta)\cap\mathfrak{D}(\Theta')$.

A colligation Δ is a quadruple $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};U)$ consisting of three Krein spaces $\mathfrak{K}$ (the inner or state space), $\mathfrak{F}$ and $\mathfrak{G}$ (the left outer or input space and the right outer or output space, respectively) and a mapping $U\in\mathbf{L}(\mathfrak{K}\oplus\mathfrak{F},\mathfrak{K}\oplus\mathfrak{G})$ (the connecting operator), which we usually write in the form of a 2×2 block matrix

$$U=\begin{pmatrix} T & F \\ G & H\end{pmatrix}:\begin{pmatrix}\mathfrak{K}\\ \mathfrak{F}\end{pmatrix}\to\begin{pmatrix}\mathfrak{K}\\ \mathfrak{G}\end{pmatrix}$$

with bounded operators T (the basic operator), F, G and H; we often write $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$. The colligation Δ is called unitary, if U is unitary and it is called closely connected if for some small neighborhood $\mathcal{N}$ of 0 contained in $\mathbb{D}$

$$\mathfrak{K}=\bigvee_{z\in\mathcal{N}}\left(\mathfrak{R}((I-zT)^{-1}F)\cup\mathfrak{R}((I-zT^{+})^{-1}G^{+})\right),$$

i.e., $\mathfrak{K}$ is the closed linear span of the elements in the union of the indicated ranges.

An operator $T\in\mathbf{L}(\mathfrak{K})$ will be called simple if there does not exist a nonzero subspace (i.e., linear subset) $\mathfrak{K}_0$ of $\mathfrak{K}$ such that $T\mathfrak{K}_0=\mathfrak{K}_0$ and $(I-T^{+}T)\mathfrak{K}_0=\{0\}$. This definition extends the one given in for instance [BDS]. Note that in the definition $\mathfrak{K}_0$ may be degenerated. It is not difficult to check that if T is simple, then so are T^{+} and $T^{*}=J_{\mathfrak{K}}T^{+}J_{\mathfrak{K}}$ for any fundamental symmetry $J_{\mathfrak{K}}$ on $\mathfrak{K}$.

LEMMA 3.1. *Let $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ be a unitary colligation. Then the following statements are equivalent :*

(*i*) *Δ is closely connected,*

(*ii*) *there does not exist a nonzero subspace $\mathfrak{K}_0$ of $\mathfrak{K}$ such that $T\mathfrak{K}_0=\mathfrak{K}_0$ and $G\mathfrak{K}_0=\{0\}$,*

(*iii*) *there does not exist a nonzero subspace $\mathfrak{K}_0$ of $\mathfrak{K}$ such that $T^{+}\mathfrak{K}_0=\mathfrak{K}_0$ and $F^{+}\mathfrak{K}_0=\{0\}$.*

Moreover, if T is simple then Δ is closely connected and the converse is true if one of the following four conditions is valid: $\mathfrak{R}(G)^{c}=\mathfrak{G}$, $\mathfrak{G}$ is a Hilbert space, $\mathfrak{R}(F^{+})^{c}=\mathfrak{F}$ or $\mathfrak{F}$ is a Hilbert space.

Proof. For the proof of the first part of the lemma we refer to the proof of Proposition 3.2 in [DLS1], which can easily be adapted from the case where $\mathfrak{F}$ and $\mathfrak{G}$ are Hilbert spaces to the case where they are Krein spaces. In order to prove the second statement, suppose that Δ is not closely connected. Then by the first part there exists a nonzero subspace $\mathfrak{K}_0$ of $\mathfrak{K}$ such that $T\mathfrak{K}_0 = \mathfrak{K}_0$ and $G\mathfrak{K}_0 = \{0\}$, whence $(I - T^+T)\mathfrak{K}_0 = G^+G\mathfrak{K}_0 = \{0\}$, so that T is not simple. Conversely, if T is not simple, then there exists a nonzero subspace $\mathfrak{K}_0$ of $\mathfrak{K}$ such that $T\mathfrak{K}_0 = \mathfrak{K}_0$ and $G^+G\mathfrak{K}_0 = (I - T^+T)\mathfrak{K}_0 = \{0\}$. If $\mathfrak{R}(G)^c = \mathfrak{G}$ or if $\mathfrak{G}$ is a Hilbert space this implies that $G\mathfrak{K}_0 = \{0\}$ and from *(ii)* it follows that Δ is not closely connected. In a similar way it can be shown that the other two conditions also imply that Δ is not closely connected.

The characteristic function Θ_Δ of a colligation $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; T, F, G, H)$ is defined by $\Theta_\Delta(z) = H + zG(I - zT)^{-1}F$. This definition stems from M.G. Krein. Clearly, $\Theta_\Delta \in \mathbf{S}(\mathfrak{F}, \mathfrak{G})$ and

$$\mathfrak{D}(\Theta_\Delta) = \{\, z \in \mathbb{D} \mid z = 0 \text{ or } 1/z \in \rho(T) \,\}.$$

LEMMA 3.2. *If* $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; T, F, G, H)$ *is unitary, then*

(i) $\quad \Theta_\Delta(z) - \Theta_\Delta(w) = (z - w)G(I - zT)^{-1}(I - wT)^{-1}F,$

(ii) $\quad I - \Theta_\Delta(w)^+\Theta_\Delta(z) = (1 - z\bar{w})F^+(I - \bar{w}T^+)^{-1}(I - zT)^{-1}F,$

(iii) $\quad I - \Theta_\Delta(z)\Theta_\Delta(w)^+ = (1 - z\bar{w})G(I - zT)^{-1}(I - \bar{w}T^+)^{-1}G^+$

and hence, for $f_j \in \mathfrak{F}$, $g_j \in \mathfrak{G}$, $j = 1, 2$,

(iv)
$$\left[\, S^m_{\Theta_\Delta}(z, w) \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \,\right] =$$
$$\left[\, (I - zT)^{-1}Ff_1 + (I - zT^+)^{-1}G^+g_1, (I - wT)^{-1}Ff_2 + (I - wT^+)^{-1}G^+g_2 \,\right].$$

The proof can be given by straightforward calculation, cf. M.S. Brodskii [Br].

COROLLARY 3.3. *Let* $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; T, F, G, H)$ *be unitary. (i) If* $\mathfrak{K}$ *is a Hilbert space or if* Δ *is closely connected, then*

$$\mathfrak{R}(F^+)^c = \bigvee_{z \in \mathcal{N}} (\, \mathfrak{R}(\Theta_\Delta(\bar{z})^+ - \Theta_\Delta(0)^+) \cup \mathfrak{R}(\, I - \Theta_\Delta(0)^+\Theta_\Delta(z)) \,),$$
$$\mathfrak{R}(G)^c = \bigvee_{z \in \mathcal{N}} (\, \mathfrak{R}(\Theta_\Delta(0) - \Theta_\Delta(z)) \cup \mathfrak{R}(\, I - \Theta_\Delta(0)\Theta_\Delta(\bar{z})^+) \,),$$

where $\mathcal{N}$ *is some neighborhood of* 0 *contained in* $\mathfrak{D}(\Theta_\Delta)$. *(ii) If* Δ *is closely connected, then*

$$\dim \mathfrak{K}_\pm = \# \begin{matrix} positive \\ negative \end{matrix} \; squares \; of \; \mathrm{S}^m_{\Theta_\Delta}(z,w),$$

where $\mathfrak{K} = \mathfrak{K}_+ + \mathfrak{K}_-$ *is any fundamental decomposition of the state space* $\mathfrak{K}$.

An operator V from $\mathfrak{F}$ to $\mathfrak{G}$ is called a weak isomorphism if it has a dense domain $\mathfrak{D}(V) \subset \mathfrak{F}$ and a dense range $\mathfrak{R}(V) \subset \mathfrak{G}$, and is isometric, that is $V^+ V = I|_{\mathfrak{D}(V)}$. A weak isomorphism is closable and hence, it may be assumed to be closed. The spaces $\mathfrak{F}$ and $\mathfrak{G}$ are called weakly isomorphic if there exists such a weak isomorphism between them. If V is a weak isomorphism from $\mathfrak{F}$ to $\mathfrak{G}$ then V is bounded (and hence, is an isomorphism between $\mathfrak{F}$ and $\mathfrak{G}$) in each of the following cases: (i) $\mathfrak{F}$ and $\mathfrak{G}$ are Pontryagin spaces and (ii) there exist fundamental symmetries $J_{\mathfrak{F}}$ and $J_{\mathfrak{G}}$ such that $VJ_{\mathfrak{F}} = J_{\mathfrak{G}} V$ on $\mathfrak{D}(V)$. For a more general result in this direction we refer to Section 7.

Two colligations $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; T, F, G, H)$ and $\Delta' = (\mathfrak{K}', \mathfrak{F}, \mathfrak{G}; T', F', G', H')$ with the same outer spaces $\mathfrak{F}$ and $\mathfrak{G}$ are called weakly isomorphic if $H = H'$ and there exists a weak isomorphism V from $\mathfrak{K}'$ to $\mathfrak{K}$ such that

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T' & F' \\ G' & H' \end{pmatrix} \text{ on } \begin{pmatrix} \mathfrak{D}(V) \\ \mathfrak{F} \end{pmatrix}.$$

If V can be extended to a unitary operator from $\mathfrak{K}'$ onto $\mathfrak{K}$, then of course, Δ and Δ' are called isomorphic or unitarily equivalent.

THEOREM 3.4. *Every* $\Theta \in \mathrm{S}(\mathfrak{F}, \mathfrak{G})$ *can be written as* $\Theta(z) = \Theta_\Delta(z)$ *with* z *in a neighborhood of* 0 *in* $\mathbb{D}$, *for some unitary colligation* $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; U)$. *Here* Δ *can be chosen to be closely connected in which case it is uniquely determined up to weak isomorphisms and*

$$\dim \mathfrak{K}_\pm = \# \begin{matrix} positive \\ negative \end{matrix} \; squares \; of \; \mathrm{S}^m_{\Theta}(z,w),$$

where $\mathfrak{K} = \mathfrak{K}_+ + \mathfrak{K}_-$ *is any fundamental decomposition of the state space* $\mathfrak{K}$.

For this theorem see T.Ya. Azizov [Az1], [Az2]. Azizov's proof starts with a result of D.Z. Arov [Ar] which states that an arbitrary operator valued mapping, holomorphic at $z=0$, is the characteristic function of a (not necessarily unitary) colligation and applies C. Davis' statement [Da] that any bounded operator has a unitary dilation in a Krein space. In [DLS2] we have stated and proved a more detailed version of Theorem 3.4 in the sense that we specified beforehand the domain of points z for which the equality $\Theta(z) = \Theta_\Delta(z)$ is to be valid. Our proof is based on Corollary 2.3, but it is quite lengthy and therefore will not be repeated here. However, in order to show how Corollary 2.3 can be applied, we shall prove Theorem 3.4 for Θ's in a special subclass of $\mathbf{S}(\mathfrak{F},\mathfrak{G})$. The assumptions make a much simpler proof possible, than the one in [DLS2] for the general case and lead to the slightly stronger conclusion that the unitary colligation can be chosen to be closely innerconnected and closely outerconnected. The reason for this is formulated in Proposition 3.5 below, which is of some interest of its own. Proposition 3.6 below is the restricted version of Theorem 3.4. Neither of the propositions will be used in the remainder of this paper.

Recall that a unitary colligation $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ is closely innerconnected if

$$\mathfrak{K}=\vee\{T^nFf \mid n\in\mathbb{N}\cup\{0\},\ f\in\mathfrak{F}\}$$

and closely outerconnected if

$$\mathfrak{K}=\vee\{T^{+n}G^+g \mid n\in\mathbb{N}\cup\{0\},\ g\in\mathfrak{G}\}.$$

Clearly, if Δ is closely innerconnected or closely outerconnected, then it is closely connected. The following result gives a sufficient condition for the converse to hold.

PROPOSITION 3.5. *Let $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ be a unitary colligation. Assume that the point $0\in\rho(T)$ and that it also belongs to the unbounded component of $\rho(T)$. If Δ is closely connected, then it is closely innerconnected and closely outerconnected. In particular, this holds if $\dim\mathfrak{K}<\infty$ and T is invertible.*

Proof. We have that

$$\bigvee_{z\in\rho(T)}\Re(T-z)^{-1}=\bigvee_{n\in\mathbf{Z}}\Re(T^n)=\bigvee_{n\geq 0}\Re(T^n).$$

The first equality is valid since $0,\infty\in\rho(T)$ and the second one follows from the fact that these points belong to the same component as this implies that T^{-1} can be approximated in the uniform topology by polynomials in T. Let $\mathcal{O}\subset\rho(T)$ be a neighborhood of 0 and ∞, which is symmetric with respect to $\partial\mathbb{D}$. We extend the definition of $\Theta_\Delta(z)$ to all values $z\in\mathbb{C}$ for which $1/z\in\rho(T)$ in the obvious way: $\Theta_\Delta(z)=H+zG(I-zT)^{-1}F$. It is easy to verify that the equalities (*ii*) and (*iii*) of Lemma 3.2 are valid for this extended Θ_Δ. They imply that $\Theta_\Delta(1/z)$ is invertible with inverse $\Theta_\Delta(1/z)^{-1}=\Theta_\Delta(\bar{z})^+$ for all $z\in\mathcal{O}$. It follows from the relation

$$(T-z)^{-1}F\Theta_\Delta(\bar{z})^+ = -(I-zT^+)^{-1}G^+, \qquad z\in\mathcal{O},$$

that

$$\bigvee_{n\geq 0}\Re(T^{+n}G^+)\subset \bigvee_{z\in\rho(T)}\Re((T-z)^{-1}F)=\bigvee_{n\geq 0}\Re(T^nF).$$

Hence, if Δ is closely connected, then it is closely innerconnected. The same reasoning applied to T^+, instead of T easily yields that then Δ is also closely outerconnected. This completes the proof.

PROPOSITION 3.6. *Assume that $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ can be extended holomorphically to a simply connected domain $\mathfrak{D}$ in the extended complex plane $\bar{\mathbb{C}}$, which is symmetric with respect to the unit circle $\partial\mathbb{D}$ and contains neighborhoods of $0,1$ and ∞, such that $\Theta(\bar{z})^{-1}=\Theta(1/z)^+$. Let ∂ be a closed smooth Jordan curve in $\mathbb{C}$ with interior, exterior denoted by $I(\partial)$, $E(\partial)$, respectively, such that $E(\partial)\cup\partial\subset\mathfrak{D}$, the points 0, 1 and ∞ belong to $E(\partial)$ and there exists a conformal mapping γ from $I(\partial)$ onto $\mathbb{D}$, which can be extended to a continuously differentiable function, also denoted by γ, from $I(\partial)\cup\partial$ onto $\mathbb{D}\cup\partial\mathbb{D}$ with $\gamma'(z)\neq 0$ for all $z\in I(\partial)\cup\partial$. Then, for all $z\in E(\partial)$, $\Theta(z)=\Theta_\Delta(z)$ for some unitary colligation Δ, that can be chosen to be closely innerconnected and closely outerconnected, in which case it is uniquely determined up to weak isomorphisms.*

REMARK. According to a theorem of Kellogg, a sufficient condition for the existence of γ is that the angle of the tangent to ∂, considered as a function of the arc length along ∂ satisfies a Lipschitz condition, see [G] Theorem 6, p. 374.

Proof. Put $\mathfrak{D}_-=\{z\in\mathbb{C}\mid 1/z\in\mathfrak{D}\}$ and $\partial_-=\{z\in\mathbb{C}\mid 1/z\in\partial\}$. Then $\mathfrak{D}_-$ and ∂_- have the same characteristics as $\mathfrak{D}$ and ∂ described in the proposition. We write γ_- for the corresponding conformal mapping from $I(\partial_-)$ onto $\mathbb{D}$. In the proof we shall consider contour integrals $\oint$ which are always over the contour ∂_- traversed in the positive direction with respect to $I(\partial_-)$ and write

$$\overline{\oint} u(z)\,\overline{dz}=\overline{\oint \overline{u(z)}\;dz}.$$

We put $\Psi(z)=\Theta(1/z)$. Then we have Cauchy's formula

$$\Psi(z)=\Psi(\infty)-\frac{1}{2\pi i}\oint\frac{\Psi(w)}{w-z}dw, \quad z\in E(\partial_-) \quad \text{(strongly)}.$$

We fix fundamental symmetries $J_{\mathfrak{F}}$ on $\mathfrak{F}$ and $J_{\mathfrak{G}}$ on $\mathfrak{G}$. When we consider $\mathfrak{F}$ and $\mathfrak{G}$

as Hilbert spaces, we mean the linear spaces $\mathfrak{F}$, $\mathfrak{G}$ endowed with the positive definite inner products $[J_{\mathfrak{F}}\cdot,\cdot]_{\mathfrak{F}}$, $[J_{\mathfrak{G}}\cdot,\cdot]_{\mathfrak{G}}$, respectively. By $\mathfrak{H}=H^2_{\mathfrak{F}}(I(\partial_-))$ we denote the Hilbert space of all functions $h:I(\partial_-)\to\mathfrak{F}$ for which $\hat{h}=h\circ\gamma_-^{-1}:\mathbb{D}\to\mathfrak{F}$ belongs to the Hardy class $H^2_{\mathfrak{F}}(\mathbb{D})$. In this class the functions can be extended to $\partial\mathbb{D}$ by their nontangential limits and hence the functions in $\mathfrak{H}$ can also be extended to ∂_-. The inner product is given by

$$(h,k)=\frac{1}{2\pi i}\oint_{\partial\mathbb{D}}[J_{\mathfrak{F}}\hat{h}(w),\hat{k}(w)]_{\mathfrak{F}}\frac{dw}{w}=$$
$$=\frac{1}{2\pi i}\oint[J_{\mathfrak{F}}h(z),k(z)]_{\mathfrak{F}}\frac{\gamma_-'(z)}{\gamma_-(z)}dz.$$

Note that because $0<c\leq|\gamma_-'/\gamma_-|\leq C<\infty$ on ∂_-, the norm corresponding to this inner product is equivalent to the norm

$$h\to(\oint[J_{\mathfrak{F}}h(z),h(z)]_{\mathfrak{F}}|dz|)^{1/2}.$$

This, the fact that Ψ is holomorphic in a neighborhood of ∂_- and the equality $\Psi(w)^+\Psi(z)=I$ when $z\bar{w}=1$, $z,w\in\partial_-$, immediately yield that the inner product $[.,.]$ defined by

$$[h,k]=\frac{1}{4\pi^2}\oint\overline{\oint}\Big[\frac{\Psi(w)^+\Psi(z)-I}{1-z\bar{w}}h(z),k(w)\Big]_{\mathfrak{F}}dz\,\overline{dw},\qquad h,k\in\mathfrak{H},$$

is bounded on $\mathfrak{H}$. We want to apply Corollary 2.3 and define the Hilbert spaces $\mathfrak{H}_1,\mathfrak{H}_2$, the inner products $[.,.]_1$, $[.,.]_2$ and the operator U_0 by

$$\mathfrak{H}_1=\mathfrak{H}\oplus\mathfrak{F},\quad \mathfrak{H}_2=\mathfrak{H}\oplus\mathfrak{G},$$

$$\left[\begin{pmatrix}h\\f_1\end{pmatrix},\begin{pmatrix}k\\f_2\end{pmatrix}\right]_1=[h,k]+[f_1,f_2]_{\mathfrak{F}},\quad \left[\begin{pmatrix}h\\g_1\end{pmatrix},\begin{pmatrix}k\\g_2\end{pmatrix}\right]_2=[h,k]+[g_1,g_2]_{\mathfrak{G}},$$

$$U_0\begin{pmatrix}h\\f\end{pmatrix}=\begin{pmatrix}zh(z)+f\\ \frac{1}{2\pi i}\oint\Psi(z)h(z)dz+\Psi(\infty)f\end{pmatrix},$$

where $h,k\in\mathfrak{H}$, $f_1,f_2,f\in\mathfrak{F}$ and $g_1,g_2\in\mathfrak{G}$. The operator U_0 maps $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ and, as one can easily verify using Cauchy's formula, is invertible with inverse given by

$$U_0^{-1}\begin{pmatrix}k\\g\end{pmatrix}=\begin{pmatrix}\frac{1}{z}k(z)+\frac{1}{z}\Psi(0)^{-1}\big(\frac{1}{2\pi i}\oint\frac{\Psi(z)}{z}k(z)dz-g\big)\\ -\Psi(0)^{-1}\big(\frac{1}{2\pi i}\oint\frac{\Psi(z)}{z}k(z)dz-g\big)\end{pmatrix}.$$

With the exception of the equality $[U_0x,U_0y]_2=[x,y]_1$, the hypotheses of

Corollary 2.3 are easily checked. The equality can be proved in a straightforward manner:

$$\left[U_0\begin{pmatrix}h\\f_1\end{pmatrix},U_0\begin{pmatrix}k\\f_2\end{pmatrix}\right]_2=$$

$$=\frac{1}{4\pi^2}\oint\overline{\oint}\left[\frac{\Psi(w)^+\Psi(z)-I}{1-z\bar{w}}(zh(z)+f_1),wk(w)+f_2\right]_{\mathfrak{F}}dz\overline{dw}+$$

$$+\left[\frac{1}{2\pi i}\oint\Psi(z)h(z)dz+\Psi(\infty)f_1,\frac{1}{2\pi i}\oint\Psi(z)k(z)dz+\Psi(\infty)f_2\right]_{\mathfrak{G}}$$

$$=\text{ 4 double integrals + 4 inner products}$$

$$=s(h,k)+s(h,f_2)+s(f_1,k)+s(f_1,f_2),$$

where $s(a,b)$ stands for the sum of the double integral and the inner product involving a and b. Using Cauchy's formula we obtain the equalities

$$s(h,k)=[h,k],\quad s(h,f_2)=s(f_1,k)=0,\quad s(f_1,f_2)=[f_1,f_2]_{\mathfrak{F}}.$$

We shall prove the second and fourth equality, as the other two can be proved in a similar way. Concerning the second relation we have

$$s(h,f_2)=\frac{1}{4\pi^2}\oint\overline{\oint}\left[\frac{\Psi(w)^+\Psi(z)-I}{1-z\bar{w}}zh(z),f_2\right]_{\mathfrak{F}}dz\overline{dw}+$$

$$+\left[\frac{1}{2\pi i}\oint\Psi(z)h(z)dz,\Psi(\infty)f_2\right]_{\mathfrak{G}}.$$

The first summand is equal to

$$\frac{1}{2\pi i}\oint\left[h(z),\frac{1}{2\pi i}\oint\frac{\Psi(z)^+\Psi(w)-I}{(\bar{z})^{-1}-w}f_2dw\right]_{\mathfrak{F}}dz$$

$$=\frac{1}{2\pi i}\oint\left[h(z),(\,(\Psi(z)^+\Psi(\bar{z}^{-1})-I)-(\Psi(z)^+\Psi(\infty)-I)\,)f_2\right]_{\mathfrak{F}}dz$$

$$=-\frac{1}{2\pi i}\oint\left[\Psi(z)h(z),\Psi(\infty)f_2\right]_{\mathfrak{G}}dz$$

and, consequently, $s(h,f_2)=0$. Also,

$$s(f_1,f_2)=\frac{1}{4\pi^2}\oint\overline{\oint}\left[\frac{\Psi(w)^+\Psi(z)-I}{1-z\bar{w}}f_1,f_2\right]_{\mathfrak{F}}dz\overline{dw}+[\Psi(\infty)f_1,\Psi(\infty)f_2]_{\mathfrak{G}}$$

and here the first summand equals

$$-\frac{1}{2\pi i}\overline{\oint}\left[\frac{1}{2\pi i}\oint\frac{\Psi(w)^+\Psi(z)-I}{(\bar{w})^{-1}-z}f_1dz,\frac{1}{w}f_2\right]_{\mathfrak{F}}\overline{dw}$$

$$= \frac{1}{2\pi i}\overline{\oint}\left[(\Psi(w)^{+}\Psi(\infty)-I)f_1, \frac{1}{w}f_2\right]_{\mathfrak{F}}\overline{dw}$$

$$= -\left[f_1, \frac{1}{2\pi i}\oint \frac{\Psi(\infty)^{+}\Psi(w)-I}{w}f_2\,dw\right]_{\mathfrak{F}}$$

$$= -\left[f_1, (\Psi(\infty)^{+}\Psi(\infty)-I)f_2\right]_{\mathfrak{G}},$$

which implies $s(f_1,f_2)=[f_1,f_2]_{\mathfrak{F}}$. Hence, the equality $[U_0x,U_0y]_2=[x,y]_1$ holds. It is not difficult to see that the Krein space associated with $\mathfrak{H}_1$ and $[.,.]_1$ ($\mathfrak{H}_2$ and $[.,.]_2$) is $\mathfrak{K}\oplus\mathfrak{F}$ ($\mathfrak{K}\oplus\mathfrak{G}$, respectively), where $\mathfrak{K}$ is the Krein space associated with $\mathfrak{H}$ and $[.,.]$. Let $U:\mathfrak{K}\oplus\mathfrak{F}\to\mathfrak{K}\oplus\mathfrak{G}$ be the continuous extension of $\hat{U}_0:\hat{\mathfrak{H}}\oplus\mathfrak{F}\to\hat{\mathfrak{H}}\oplus\mathfrak{G}$, which exists and is unitary by Corollary 2.3. Then $\Delta=(\mathfrak{K};\mathfrak{F},\mathfrak{G};U)$ is a unitary colligation and we claim that $\Theta(z)=\Theta_{\Delta}(z)$, $z\in E(\partial_-)$. Indeed, writing

$$U_0=\begin{pmatrix} T_0 & F_0 \\ G_0 & H_0 \end{pmatrix}:\begin{pmatrix}\mathfrak{H}\\ \mathfrak{F}\end{pmatrix}\to\begin{pmatrix}\mathfrak{H}\\ \mathfrak{G}\end{pmatrix},$$

where $(T_0h)(w)=wh(w)$, $(F_0f)(w)=f$, $G_0h=\frac{1}{2\pi i}\oint\Psi(w)h(w)\,dw$ and $H_0f=\Psi(\infty)f$, we have that for $z\in E(\partial_-)$

$$(H_0+zG_0(I-zT_0)^{-1}F_0)f=\Psi(\infty)f+\frac{z}{2\pi i}\oint\Psi(w)\,\frac{1}{1-zw}f\,dw=$$

$$=\Psi(\infty)f+\frac{1}{2\pi i}\oint\frac{\Psi(w)}{z^{-1}-w}f\,dw=\Psi(\infty)f+(\Psi(1/z)-\Psi(\infty))f=\Theta(z)f$$

and from this the claim can be proved. We leave the details to the reader. It is easy to check that $T_0^nF_0f=w^nf$, $f\in\mathfrak{F}$, which implies that Δ is closely innerconnected. To see this, it is sufficient to show that the polynomials with coefficients in $\mathfrak{F}$ are dense in $\mathfrak{H}=H^2_{\mathfrak{F}}(I(\partial_-))$. Since these polynomials are dense in $H^2_{\mathfrak{F}}(\mathbb{D})$, it follows that when composed with γ_- they form a dense set in $\mathfrak{H}$. Hence, it suffices to prove that for each $n\in\mathbb{N}$ and $f\in\mathfrak{F}$ the function $\gamma_-(z)^nf$ can be approximated in $\mathfrak{H}$ by polynomials in z with coefficients in $\mathfrak{F}$. But this follows from Mergelyan's theorem (see, for instance, [R]), which implies that $\gamma_-(z)^n$ can, in fact, be approximated by polynomials in z, uniformly on $I(\partial_-)\cup\partial$. Hence, Δ is closely innerconnected. Finally, writing U_0^{-1} as

$$U_0^{-1}=\begin{pmatrix}\tilde{T}_0 & \tilde{G}_0\\ \tilde{F}_0 & \tilde{H}_0\end{pmatrix}:\begin{pmatrix}\mathfrak{H}\\ \mathfrak{G}\end{pmatrix}\to\begin{pmatrix}\mathfrak{H}\\ \mathfrak{F}\end{pmatrix},$$

we have that the linear span $\bigvee_{j=1}^{n}\Re(\tilde{T}_0^j\tilde{G}_0)$ is equal to the linear space

$\{\sum_{j=1}^{n} z^{-j} f_j \,|\, f_j \in \mathfrak{F}\} \subset \mathfrak{H}$ and the above argument (in the end applied to the function $(1/\gamma_-(z))^n/z$ instead of $\gamma_-(z)^n$) gives that Δ is closely outerconnected. We leave to the reader the proof of the uniqueness of Δ up to weak isomorphisms.

We thank Prof. B.L.J. Braaksma for pointing out the above application of Mergelyan's theorem. By similar methods one can obtain Caratheodory type representations of holomorphic operator functions, see, e.g., [DLS3].

4. CHARACTERISTIC FUNCTIONS OF BOUNDED OPERATORS IN KREIN SPACES

The definition of the characteristic function of a bounded operator in a Hilbert space goes at least back to M.S. Livšic and V.P. Potapov [LP], A.V. Štraus [St] and Yu.L. Smul'jan [Sm]. In this section we extend it to a bounded operator on a Krein space. The ideas leading to this extension are taken mainly from [ACG], where Arsene, Constantinescu and Gheondeaconsider bounded operators acting from one space to another. For our purpose, however, it suffices to consider a bounded operator T from a Krein space $(\mathfrak{K}, [.,.])$ to itself. Relative to a fixed fundamental symmetry $J_{\mathfrak{K}}$ we define the operators

$$J_T = \operatorname{sgn}(J_{\mathfrak{K}} - T^* J_{\mathfrak{K}} T), \quad D_T = |J_{\mathfrak{K}} - T^* J_{\mathfrak{K}} T|^{1/2},$$

computed using the functional calculus on the Hilbert space $(\mathfrak{K}, [J_{\mathfrak{K}}.,.])$, and we set $\mathfrak{D}_T = \Re(D_T)^c$. Note that $\mathfrak{D}_T$ endowed with the inner product $[J_{\mathfrak{K}} J_T.,.]$ is a Krein space and when we refer to $\mathfrak{D}_T$ as a Krein space it is with respect to this inner product.

THEOREM 4.1. *Let $\mathfrak{K}$ be a Krein space, $J_{\mathfrak{K}}$ a fundamental symmetry on $\mathfrak{K}$ and $T \in \mathrm{L}(\mathfrak{K})$. Then*

(*i*) *there exists a uniquely determined operator $L_T \in \mathrm{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ such that $D_{T^*} L_T = T J_{\mathfrak{K}} D_T$ on $\mathfrak{D}_T$,*

(*ii*) *the operator U_T with decomposition*

$$U_T = \begin{pmatrix} T & D_{T^*}|_{\mathfrak{D}_{T^*}} \\ D_T & -J_T L_{T^*} \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_{T^*} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ \mathfrak{D}_T \end{pmatrix}$$

defines a unitary colligation $\Delta_T = (\mathfrak{K}, \mathfrak{D}_{T^*}, \mathfrak{D}_T; U_T)$ and

(*iii*) *Δ_T is closely connected if and only if T is simple.*

Part (*iii*) of the Theorem 4.1 is a consequence of the foregoing parts and the last statement in Lemma 3.1 as $\mathfrak{R}(D_T)^c = \mathfrak{D}_T$. Parts (i) and (ii) are copied from [ACG], Proposition 4.1 and Corollary 4.5, respectively. Their proof of part (i) applies Corollary 2.2 of this note and in order to show this we briefly repeat it. Take in Corollary 2.2 $\mathfrak{H} = \mathfrak{D}_{T^*}$, provided with the Hilbert inner product $(x,y) = [J_{\mathfrak{R}}x,y]$, $x,y \in \mathfrak{D}_{T^*}$, $[x,y] = (D_{T^*}^2 x,y)$ and $S_0 = J_{T^*}J_{\mathfrak{R}}TJ_TJ_{\mathfrak{R}}T^*|_{\mathfrak{D}_{T^*}}$. Then the hypotheses of Corollary 2.2 are satisfied and, moreover, $[.,.]$ is positive definite. It follows that there exists a constant $C \geq 0$ such that $[Sx,x] \leq C[x,x]$ for all $x \in \mathfrak{K}$, the Hilbert space associated with $\mathfrak{H}$ and $[.,.]$. Hence,

$$(TJ_{\mathfrak{R}}D_T)(TJ_{\mathfrak{R}}D_T)^* = D_{T^*}^2 S_0 \leq CD_{T^*}^2 \text{ on } \mathfrak{D}_{T^*}.$$

The Douglas factorization theorem (see [Fu] p. 124) now implies the existence of L_T with the desired property; its uniqueness follows from the injectivity of D_{T^*}.

We refer to [ACG] for special cases concerning the link operator L_T and the various properties of the operators J_T, D_T and L_T such as

$$L_T^+ = L_{T^*}, \quad D_T^+ = J_{\mathfrak{R}}D_TJ_T, \quad D_T^+J_TL_{T^*} = T^+D_{T^*} \text{ on } \mathfrak{D}_{T^*},$$

if D_T is considered as a (bounded) mapping from the Krein space $\mathfrak{K}$ to the Krein space $\mathfrak{D}_T$.

Theorem 4.1 gives rise to the following definition. We define the characteristic function Θ_T of $T \in \mathbf{L}(\mathfrak{K})$ relative to a fundamental symmetry $J_{\mathfrak{R}}$ to be the mapping

$$\Theta_T(z) = \Theta_{\Delta_T}(z) = -J_TL_{T^*} + zD_T(I - zT)^{-1}D_{T^*}|_{\mathfrak{D}_{T^*}}, \qquad z \in \mathfrak{D}(\Theta_{\Delta_T}),$$

i.e., the characteristic function of the unitary colligation Δ_T. This notion reduces to the usual one, if $\mathfrak{K}$ is a Hilbert space, for then $J_{\mathfrak{R}} = I$, $L_T = T$ and

$$\Theta_T(z) = -T^*J_{T^*} + zD_T(I - zT)^{-1}D_{T^*}|_{\mathfrak{D}_{T^*}}$$

with $J_T = \operatorname{sgn}(I - T^*T)$ and $D_T = |I - T^*T|^{1/2}$.

5. Characteristic functions of bounded operators in Krein spaces (continued)

If in a unitary colligation $\Delta = (\mathfrak{K}, \mathfrak{F}, \mathfrak{G}; U)$ in which U is decomposed as in (1.1) and the null spaces $\nu(F)$ and $\nu(G^+)$ of F and G^+ are orthocomplemented in $\mathfrak{F}$ and $\mathfrak{G}$, then U has the 3x3 block matrix representation

$$U=\begin{pmatrix} T & F_1 & 0\\ G_1 & H_1 & 0\\ 0 & 0 & H_2\end{pmatrix}:\begin{pmatrix}\mathfrak{K}\\ \mathfrak{R}(F^+)^c\\ \nu(F)\end{pmatrix}\to\begin{pmatrix}\mathfrak{K}\\ \mathfrak{R}(G)\\ \nu(G^+)\end{pmatrix},$$

in which $\begin{pmatrix} T & F_1\\ G_1 & H_1\end{pmatrix}$ and H_2 are unitary. In this section we want to investigate some sort of equivalence between Δ and the unitary colligation Δ_T corresponding to its basic operator T. A necessary condition for this, even when $\nu(F)$ and $\nu(G^+)$ are not assumed to be orthocomplemented, is that they are equal to $\{0\}$, or, equivalently, that $\mathfrak{R}(F^+)^c=\mathfrak{F}$ and $\mathfrak{R}(G)^c=\mathfrak{G}$, as the corresponding entries in U_T have these properties. With the following definition of equivalence these conditions turn out to be sufficient as well. We shall say that two colligations $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ and $\Delta'=(\mathfrak{K},\mathfrak{F}',\mathfrak{G}';T,F',G',H')$ with the same state space $\mathfrak{K}$ and basic operator T coincide weakly if there are two weak isomorphisms V from $\mathfrak{F}'$ to $\mathfrak{F}$ and W from $\mathfrak{G}'$ to $\mathfrak{G}$ such that

$$\begin{pmatrix} T & F\\ G & H\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & V\end{pmatrix}=\begin{pmatrix} I & 0\\ 0 & W\end{pmatrix}\begin{pmatrix} T & F'\\ G' & H'\end{pmatrix} \text{ on } \begin{pmatrix}\mathfrak{K}\\ \mathfrak{D}(V)\end{pmatrix}.$$

Similarly, $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ and $\Theta'\in\mathbf{S}(\mathfrak{F}',\mathfrak{G}')$ are said to coincide weakly if for some V and W as above $\Theta(z)V=W\Theta'(z)$ for z in some neighborhood of 0 contained in $\mathfrak{D}(\Theta)\cap\mathfrak{D}(\Theta')$. Clearly, if two colligations Δ and Δ' coincide weakly, then so do their characteristic function. If V and W are continuous, then they are unitary operators from $\mathfrak{F}'$ onto $\mathfrak{F}$ and $\mathfrak{G}'$ onto $\mathfrak{G}$, respectively, and Δ and Δ' as well as Θ_Δ and $\Theta_{\Delta'}$ coincide. A sufficient condition to ensure boundedness of these operators will be formulated and proved in Section 7.

Proposition 5.1. *Let $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ be a unitary colligation and let $J_{\mathfrak{K}}$ be a fundamental symmetry on its state space $\mathfrak{K}$. Let Δ_T be the unitary colligation defined for the basic operator T of Δ relative to $J_{\mathfrak{K}}$. Then Δ and Δ_T coincide weakly if and only if $\mathfrak{R}(F^+)^c=\mathfrak{F}$ and $\mathfrak{R}(G)^c=\mathfrak{G}$.*

Proof. Assume $\mathfrak{R}(F^+)^c=\mathfrak{F}$, $\mathfrak{R}(G)^c=\mathfrak{G}$ and define $W:\mathfrak{R}(D_T)\subset\mathfrak{D}_T\to\mathfrak{R}(G)\subset\mathfrak{G}$ by $WD_Tx=Gx$, $x\in\mathfrak{K}$. To see that W is a well defined weak isomorphism we first note that, as Δ is unitary, $I-T^+T=G^+G$ and hence we have that for $x',y'\in\mathfrak{K}$

$$[Gx',Gy']_{\mathfrak{G}}=[(I-T^+T)x',y']_{\mathfrak{K}}=$$

$$=[J_{\mathfrak{K}}J_TD_T^2x',y']_{\mathfrak{K}}=(J_TD_T^2x',y')_{\mathfrak{K}}=(J_TD_Tx',D_Ty')_{\mathfrak{D}_T}.$$

From the density of the ranges $\Re(G)$ and $\Re(D_T)$ in $\mathfrak{G}$ and $\mathfrak{D}_T$, respectively, it follows that the null spaces of G and D_T coincide and this implies that W is well defined. If $x,y\in\Re(D_T)$ and $x=D_Tx'$ and $y=D_Ty'$ for some $x',y'\in\mathfrak{K}$, then the above equalities show that $[Wx,Wy]_{\mathfrak{G}}=[x,y]_{\mathfrak{D}_T}$, i.e., W is isometric. Since $\mathfrak{D}(W)=\Re(D_T)$ and $\Re(W)=\Re(G)$, W is a weak isomorphism. Similarly, it can be shown that the mapping $V:\Re(D_{T^*})\subset\mathfrak{D}_{T^*}\to\Re(F^+)\subset\mathfrak{F}$ defined by

$$VJ_{T^*}D_{T^*}J_{\mathfrak{K}}x=F^+x,\quad x\in\mathfrak{K},$$

is a well defined isometry with a dense domain and, by assumption, a dense range. Therefore, V is a weak isomorphism between $\mathfrak{D}_{T^*}$ and $\mathfrak{F}$. From the defining relation of V and the fact that the Krein space adjoint of

$$D_{T^*}|_{\mathfrak{D}_{T^*}}:\mathfrak{D}_{T^*}\to\mathfrak{K}$$

is given by

$$(D_{T^*}|_{\mathfrak{D}_{T^*}})^+=J_{T^*}D_{T^*}J_{\mathfrak{K}},$$

it easily follows that $FV=D_{T^*}$ on $\mathfrak{D}(V)$. It remains to prove that $HV=-WJ_TL_{T^*}$ on $\mathfrak{D}(V)$. As Δ is unitary $F^+T=-H^+G$ and therefore,

$$V(J_TL_{T^*})^+D_T=V(D_T^+J_TL_{T^*})^+=V(T^+D_{T^*}|_{\mathfrak{D}_{T^*}})^+=$$

$$=VJ_{T^*}D_{T^*}J_{\mathfrak{K}}T=F^+T=-H^+G=-H^+WD_T.$$

This implies that $V(J_TL_{T^*})^+=-H^+W$ on $\Re(D_T)$ and therefore, $(J_TL_{T^*})^+W^+=-V^+H^+$ on $\Re(W)$. Taking adjoints we obtain the desired equality. The proof of the converse is left to the reader.

We now come to the main result of this paper.

THEOREM 5.2. *Let* $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$. *If*

$$(5.1)\quad\begin{cases}\mathfrak{F}=\bigvee_{z\in N}\left(\Re(\Theta(\bar z)^+-\Theta(0)^+)\cup\Re(I-\Theta(0)^+\Theta(z))\right),\\[2ex] \mathfrak{G}=\bigvee_{z\in N}\left(\Re(\Theta(0)-\Theta(\bar z))\cup\Re(I-\Theta(0)\Theta(\bar z)^+)\right),\end{cases}$$

where N is some neighborhood of 0 *contained in $\mathfrak{D}(\Theta)$, then Θ coincides weakly with a Θ_T for some bounded operator T on a Krein space $\mathfrak{K}$. Here T can be chosen to be simple in which case*

$$\dim\mathfrak{K}_\pm=\#\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\ \textit{squares of}\ S_\Theta^m(z,w),$$

where $\mathfrak{K}=\mathfrak{K}_++\mathfrak{K}_-$ is any fundamental decomposition of $\mathfrak{K}$. If Θ coincides weakly *with Θ_T and if T is simple or $\mathfrak{K}$ is a Hilbert space, then the equalities* (5.1) *are valid.*

Proof. According to Theorem 3.4 Θ can be represented as the characteristic function Θ_Δ of a closely connected unitary colligation Δ. By Corollary 3.3 (i) and (5.1) $\mathfrak{R}(F^+)^c=\mathfrak{F}$ and $\mathfrak{R}(G)^c=\mathfrak{G}$. From Proposition 5.1 it follows that Δ and Δ_T coincide weakly, where T is the basic operator of Δ and hence, so do $\Theta=\Theta_\Delta$ and Θ_{Δ_T}. Moreover, Δ_T is closely connected and therefore, T is simple, see Theorem 4.1 (iii). Now assume that V and W are weak isomorphisms such that $\Theta V=W\Theta_T$, where Θ_T is the characteristic function of some simple, bounded operator T on a Krein space $\mathfrak{K}$ relative to a fundamental symmetry $J_{\mathfrak{K}}$. Then

$$\mathsf{S}^m_\Theta(z,w)=\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}\mathsf{S}^m_{\Theta_{\Delta_T}}(z,w)\begin{pmatrix} V^+ & 0 \\ 0 & W^+ \end{pmatrix} \text{ on } \begin{pmatrix} \mathfrak{R}(V) \\ \mathfrak{R}(W) \end{pmatrix}$$

and the formula concerning $\dim \mathfrak{K}_\pm$ now easily follows from Theorem 4.1 (iii) and Corollary 3.3 (ii). The last part of the theorem can be proved in the same vein.

6. J.A. Ball's characterization of characteristic functions of bounded operators in Hilbert spaces

The following representation theorem is due to Ball, see [Ba]. Its formulation in terms of Krein spaces stems from B.W. McEnnis, see [Mc].

THEOREM 6.1. *The mapping $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ coincides with $\Theta_T\in\mathbf{S}(\mathfrak{D}_{T^*},\mathfrak{D}_T)$ for some bounded operator T on a Hilbert space if and only if*

(i) *$I-\Theta(0)^+\Theta(0)$ is injective on $\mathfrak{F}$ and commutes with some fundamental symmetry $J_{\mathfrak{F}}$ on $\mathfrak{F}$,*

(ii) *$I-\Theta(0)\Theta(0)^+$ is injective on $\mathfrak{G}$ and commutes with some fundamental symmetry $J_{\mathfrak{G}}$ on $\mathfrak{G}$ and*

(iii) *the kernel $\mathsf{S}^m_\Theta(z,w)$ is positive semidefinite.*

Theorem 6.1 is an extension of a result of D.N. Clark [C] who considered the case where $\Theta(0)$ and hence T are invertible. V.M. Brodskii, I.C. Gohberg and M.G. Krein treated this same case in [BGK]. For the case where $\Theta(0)$ and hence T are contractions we refer to the monograph [Sz.F]. Before we show how Theorem 6.1 can be deduced from Theorem 5.2, we want to make the following remarks.

REMARKS 1. McEnnis proved in [Mc] that Theorem 6.1 remains valid if property (iii) is replaced by: the kernels

$$\frac{I-\Theta(w)^{+}\Theta(z)}{1-\bar{w}z} \text{ and } \frac{I-\Theta(\bar{w})\Theta(\bar{z})^{+}}{1-\bar{w}z}$$

(with values in $\mathbf{L}(\mathfrak{F})$ and $\mathbf{L}(\mathfrak{G})$, respectively) are positive semidefinite. Note that these kernels are the elements on the main diagonal of the matrix kernel $\mathsf{S}_{\Theta}^{m}(z,w)$. In case $\mathfrak{F}$ and $\mathfrak{G}$ are Hilbert spaces it is known that $\mathsf{S}_{\Theta}^{m}(z,w)$ has κ negative squares if and only if one of the kernels above has κ negative squares, see, e.g., [DLS1] Theorem 6.1.

2. If $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ has the property that $I-\Theta(0)^{+}\Theta(0)$ is positive definite on $\mathfrak{F}$ and commutes with a fundamental symmetry $J_{\mathfrak{F}}$ on $\mathfrak{F}$, then the operators $J_{\mathfrak{F}}$ and $J=\operatorname{sgn}(I-\Theta(0)^{+}\Theta(0))$ are equal, where the latter is computed via the functional calculus on $\mathfrak{F}$ with the Hilbert space inner product $[J_{\mathfrak{F}}\cdot,\cdot]$. Indeed, McEnnis showed that J too is a fundamental symmetry on $\mathfrak{F}$ and as it commutes with $J_{\mathfrak{F}}$ the equality $J=J_{\mathfrak{F}}$ follows, cf. [Mc] p. 165. Hence, if $\Theta\in\mathbf{S}(\mathfrak{F},\mathfrak{G})$ has the properties (i)–(iii) of Theorem 6.1, then $J_{\mathfrak{F}}=\operatorname{sgn}(I-\Theta(0)^{+}\Theta(0))$ and $J_{\mathfrak{G}}=\operatorname{sgn}(I-\Theta(0)\Theta(0)^{+})$.

3. McEnnis also proved that the isomorphisms $V\in\mathbf{L}(\mathfrak{D}_{T^*},\mathfrak{F})$ and $W\in\mathbf{L}(\mathfrak{D}_{T},\mathfrak{G})$ such that $\Theta(z)V=W\Theta_T(z)$ for z near 0 are not only unitary with respect to the indefinite inner products, but are also unitary with respect to the positive definite inner products induced by the symmetries J_{T^*}, J_T, $J_{\mathfrak{F}}$ and $J_{\mathfrak{G}}$. This now follows easily from the previous remark. E.g., $\Theta(0)V=W\Theta_T(0)$ implies that

$$(I-\Theta(0)^{+}\Theta(0))V=V(I-T^{*}T)|_{\mathfrak{D}_{T^*}}$$

(see the argument in the proof of Theorem 6.1 below), whence

$$J_{\mathfrak{F}}V=\operatorname{sgn}(I-\Theta(0)^{+}\Theta(0))V=V\operatorname{sgn}(I-T^{*}T)|_{\mathfrak{D}_{T^*}}=VJ_{T^*}$$

and therefore, V is unitary with respect to the positive definite inner products.

Proof of Theorem 6.1. If Θ coincides with Θ_T, then it has the properties (i)–(iii), as $\Theta_T\in\mathbf{S}(\mathfrak{D}_{T^*},\mathfrak{D}_T)$ has these properties. This is easy to verify and we skip the details. It remains to prove the converse. Assume Θ satisfies (i)–(iii), then (5.1) of Theorem 5.2 is valid and Θ coincides weakly with a Θ_T where T is a simple, bounded operator on a Krein space $\mathfrak{K}$:

there exist weak isomorphisms V from $\mathfrak{D}_{T^*}$ to $\mathfrak{F}$ and W from $\mathfrak{D}_T$ to $\mathfrak{G}$ such that $\Theta(z)V = W\Theta_T(z)$ on $\mathfrak{D}(V)$ for z near 0. The proof of Theorem 6.1 is complete when we have shown that V and W are bounded. For, then they are unitary operators and Θ and Θ_T coincide. It suffices to prove the continuity of V, that of W can be established in a similar way. From $\Theta(0)V = W\Theta_T(0)$ we obtain on account of Theorem 4.1(*ii*), that on $\mathfrak{D}(V)$

$$V^+((I-\Theta(0)^+\Theta(0))V = I-(W\Theta_T(0))^+W\Theta_T(0) = I-\Theta_T(0)^+\Theta_T(0)$$

$$= (D_{T^*}|_{\mathfrak{D}_{T^*}})^+(D_{T^*}|_{\mathfrak{D}_{T^*}}) = V^+VJ_{T^*}D_{T^*}J_{\mathfrak{K}}(D_{T^*}|_{\mathfrak{D}_{T^*}}).$$

Hence, since V^+ is injective, we have the intertwining relation $AV=VB$, where we have put $A=I-\Theta(0)^+\Theta(0)$ and $B=J_{T^*}D_{T^*}J_{\mathfrak{K}}(D_{T^*}|_{\mathfrak{D}_{T^*}})$. From the assumptions (*i*) and (*iii*) in Theorem 6.1 concerning Θ it can easily be verified that A has the following properties:

> $A\in\mathbf{L}(\mathfrak{F})$, A is positive definite and commutes with a fundamental symmetry $J_{\mathfrak{F}}$ on $\mathfrak{F}$.

Assumption (*iii*) in Theorem 6.1 implies that $\mathfrak{K}$ is a Hilbert space, hence $J_{\mathfrak{K}} = I$ and $B = J_{T^*}(D_{T^*}|_{\mathfrak{D}_{T^*}})^2 = (I-TT^*)|_{\mathfrak{D}_{T^*}}$. Hence, it follows that B has the same properties as A:

> $B\in\mathbf{L}(\mathfrak{D}_{T^*})$, B is positive definite on the Krein space $\mathfrak{D}_{T^*}$ and commutes with the fundamental symmetry J_{T^*} on $\mathfrak{D}_{T^*}$.

The following proposition with $\mathfrak{G} = \mathfrak{D}_{T^*}$ implies that V has the desired property. In the next section we treat a generalization of this result, see Theorem 7.1.

PROPOSITION 6.2. *Let $V:\mathfrak{G}\to\mathfrak{F}$ be a weak isomorphism between the Krein spaces $\mathfrak{G}$ and $\mathfrak{F}$. Assume there exist positive definite operators $A\in\mathbf{L}(\mathfrak{F})$ and $B\in\mathbf{L}(\mathfrak{G})$ which commute with a fundamental symmetry on $\mathfrak{F}$ and on $\mathfrak{G}$, respectively, such that $AV = VB$ on $\mathfrak{D}(V)$. Then V is a unitary operator from $\mathfrak{G}$ onto $\mathfrak{F}$.*

Proof. The positive definiteness implies in particular that both operators are selfadjoint, injective and have dense ranges. It follows that the spaces $B\mathfrak{D}(V)$ and $A\mathfrak{R}(V)$ are dense in $\mathfrak{G}$ and $\mathfrak{F}$, respectively, and that V maps $B\mathfrak{D}(V)$ isometrically and bijectively onto $A\mathfrak{R}(V)$. Let $J_{\mathfrak{G}}$ ($J_{\mathfrak{F}}$) be the fundamental symmetry on $\mathfrak{G}$ ($\mathfrak{F}$) which commutes with B (A, respectively). Recall that in $(\mathfrak{G},[.,.]_{\mathfrak{G}})$ $((\mathfrak{F},[.,.]_{\mathfrak{F}}))$ we have denoted by $(.,.)_{\mathfrak{G}}$ $((.,.)_{\mathfrak{F}})$ and $\|\ \|_{\mathfrak{G}}$

$(\|\ \|_{\mathfrak{F}})$ the Hilbert space inner products and corresponding norms on $\mathfrak{G}$ $(\mathfrak{F})$ generated by $J_{\mathfrak{G}}$ ($J_{\mathfrak{F}}$, respectively). From the inequality

$$|[B^{-1}y,x]_{\mathfrak{G}}|^2=|(J_{\mathfrak{G}}B^{-1}y,x)_{\mathfrak{G}}|^2\leq(J_{\mathfrak{G}}B^{-1}y,y)_{\mathfrak{G}}(J_{\mathfrak{G}}B^{-1}x,x)_{\mathfrak{G}}$$

with $y=BJ_{\mathfrak{G}}x$, one easily obtains the inequality

$$\|x\|_{\mathfrak{G}}^2\leq\|J_{\mathfrak{G}}B\|_{\mathfrak{G}}[B^{-1}x,x]_{\mathfrak{G}},\qquad x\in\mathfrak{R}(B).$$

We want to apply Corollary 2.3 and define $\mathfrak{H}_1$ as the Hilbert space completion of $\mathfrak{R}(B)$ with respect to the positive definite inner product $(x,y)_1=[B^{-1}x,y]_{\mathfrak{G}}$ and $\mathfrak{H}_2$ as the Hilbert space completion of $\mathfrak{R}(A)$ with respect to the positive definite inner product $(u,v)_2=[A^{-1}u,v]_{\mathfrak{F}}$. The above inequality and the relation

$$[x,y]_{\mathfrak{G}}=(Bx,y)_1,\qquad x\in\mathfrak{G},\quad y\in\mathfrak{H}_1,$$

imply that $\mathfrak{H}_1$ can be identified with a linear manifold in $\mathfrak{G}$, that $B\mathfrak{D}(V)$ is dense in $\mathfrak{H}_1$ and that the inner product $[.,.]_1$, defined as the restriction of $[.,.]_{\mathfrak{G}}$ to $\mathfrak{H}_1$ is bounded on $\mathfrak{H}_1$ with B as the Gram operator. Similarly, it can be shown that $\mathfrak{H}_2$ can be identified with a linear manifold in $\mathfrak{F}$, that $A\mathfrak{R}(V)$ is dense in $\mathfrak{H}_2$ and that the restriction $[.,.]_2$ of $[.,.]_{\mathfrak{F}}$ on $\mathfrak{H}_2$ is a bounded inner product on $\mathfrak{H}_2$ with A as the Gram operator. Since for $x,y\in B\mathfrak{D}(V)\subset\mathfrak{D}(V)$ we have that $Vx,Vy\in A\mathfrak{R}(V)$ and

$$(Vx,Vy)_2=[A^{-1}Vx,Vy]_{\mathfrak{F}}=[VB^{-1}x,Vy]_{\mathfrak{F}}=[B^{-1}x,y]_{\mathfrak{G}}=(x,y)_1.$$

V on $B\mathfrak{D}(V)$ can be extended by continuity to a unitary and hence bijective mapping $U_0\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ and $[Vx,Vy]_{\mathfrak{F}}=[x,y]_{\mathfrak{G}}$ implies that

$$[U_0x,U_0y]_2=[x,y]_1,\qquad x,y\in\mathfrak{H}_1.$$

We are now in a position to apply Corollary 2.3. Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ and $U\in\mathbf{L}(\mathfrak{K}_1,\mathfrak{K}_2)$ be as in its conclusion. Let $P_\pm$ be the orthogonal projection of $\mathfrak{G}$ onto $\mathfrak{G}_\pm=\mathfrak{R}(I\mp J_{\mathfrak{G}})$. Then, since B commutes with $J_{\mathfrak{G}}$, we have that $P_\pm\mathfrak{H}_2\subset\mathfrak{H}_2$. It follows that we can identify $\mathfrak{K}_2$ with $\mathfrak{G}$ and, similarly, $\mathfrak{K}_1$ with $\mathfrak{F}$. Finally, it can be verified that $V=U$, which completes the proof.

7. A SUFFICIENT CONDITION FOR THE CONTINUITY OF A WEAK ISOMORPHISM

We begin by recalling some facts from [L]. Let S be a densely defined selfadjoint operator with $\rho(S)\neq\emptyset$ on a Krein space $(\mathfrak{K},[.,.])$ and suppose that the form $[S.,.]$ has a finite number of negative squares on the domain $\mathfrak{D}(S)$.

Then S is definitizable, i.e., $\rho(S)\neq\emptyset$ and there exists a polynomial p with real coefficients such that $[p(S)x,x]\geq 0$ for all $x\in\mathfrak{D}(S^k)$ where k is the degree of p. It follows that S has a spectral function E on $\mathfrak{K}$. A critical point t of E in $\bar{\mathbb{R}}$ (the one point compactification of $\mathbb{R}$), which is also called a critical point of S, is said to be regular if there exists an open neighborhood $\Delta_0\subset\bar{\mathbb{R}}$ of t, in which t is the only critical point, such that the projections $E(\Delta)$, $\bar{\Delta}\subset\Delta_0\backslash\{t\}$, are uniformly bounded. A critical point which is not regular is called singular. The set of singular critical points of S will be denoted by $c_s(S)$.

Let us say that a bounded operator A on a Krein space $(\mathfrak{K},[.,.])$ has property P_κ, if

A is injective, $A=A^+$, the form $[A.,.]$ has κ negative squares and $0\notin c_s(A)$.

If $A\in\mathbf{L}(\mathfrak{K})$ has property P_κ, then $S=A^{-1}$ is a densely defined selfadjoint operator, the form $[S.,.]$ has κ negative squares on $\mathfrak{D}(S)$ and $\infty\notin c_s(S)$. It can be shown that $A\in\mathbf{L}(\mathfrak{K})$ has property P_0 if and only if $A=A^+$, A is positive definite and commutes with some fundamental symmetry on $\mathfrak{K}$, cf. [Cu]. The latter properties are precisely the ones we arrived at at the end of the previous section.

The main theorem of this section is as follows.

THEOREM 7.1. *Let V be a weak isomorphism from the Krein space $\mathfrak{G}$ to the Krein space $\mathfrak{F}$. Suppose that there exist operators $A\in\mathbf{L}(\mathfrak{F})$ and $B\in\mathbf{L}(\mathfrak{G})$ having properties P_κ and $\mathsf{P}_{\kappa'}$, respectively, such that $AV=VB$ on $\mathfrak{D}(V)$. Then V is a unitary operator from $\mathfrak{G}$ onto $\mathfrak{F}$ and $\kappa=\kappa'$.*

In order to prove the theorem we need the following preliminary results. If a subspace $\mathfrak{L}$ of a Krein space $(\mathfrak{K},[.,.])$ is nondegenerate and decomposable, then there exists a fundamental decomposition

(7.1) $\quad \mathfrak{L}=\mathfrak{L}_+ + \mathfrak{L}_-$, orthogonal direct sum,

in which $\mathfrak{L}_+$, $\mathfrak{L}_-$ are positive, negative subspaces of $\mathfrak{K}$, respectively. Hence $\mathfrak{L}$ has a fundamental symmetry $J_\mathfrak{L}$. The topology on $\mathfrak{L}$ induced by the norm $[J_\mathfrak{L}x,x]^{1/2}$ is called a decomposition majorant on $\mathfrak{L}$. Recall that in the case $\mathfrak{L}=\mathfrak{K}$ the decomposition majorant is unique, i.e., independent of the fundamental symmetries on $\mathfrak{K}$, and we shall refer to this topology as the norm topology on $\mathfrak{K}$. A subspace $\mathfrak{L}$ of $(\mathfrak{K},[.,.])$ is termed uniformly decomposable if it admits a decomposition of the form (7.1), such that $\mathfrak{L}_+$, $\mathfrak{L}_-$ are uniformly positive, uniformly negative subspaces of $\mathfrak{K}$, respectively.

LEMMA 7.2. *If $\mathfrak{L}$ is a nondegenerate decomposable subspace of a Krein space $\mathfrak{K}$, then the following statements are equivalent:*

(*i*) *There exists a fundamental symmetry J on $\mathfrak{K}$ such that $J\mathfrak{L}\subset\mathfrak{L}$.*

(*ii*) *The subspace $\mathfrak{L}$ is uniformly decomposable.*

(*iii*) *There is a decomposition majorant on $(\mathfrak{L},[.,.])$ which is equivalent to the norm topology of $\mathfrak{K}$ restricted to $\mathfrak{L}$.*

The implications $(i)\Rightarrow(ii)\Leftrightarrow(iii)$ are easy to prove. The implication $(ii)\Rightarrow(i)$ follows from, e.g., [An] Chapter 1, see also [Bo] Theorem V.9.1. If $\mathfrak{L}$ is a nondegenerate, decomposable and dense subspace of $\mathfrak{K}$ and satisfies (iii) of Lemma 7.2, then $\mathfrak{K}$ is the Krein space completion of $\mathfrak{L}$ with respect to the decomposition majorant referred to in (iii).

Now we recall and supplement some results from [Cu], where criteria are given for the regularity of the crititcal point ∞ of a definitizable (densely defined, selfadjoint) operator, here denoted by S, on a Krein space $(\mathfrak{K},[.,.])$. In these criteria the domain $\mathfrak{D}(S)$ and the set $\mathfrak{D}[JS]$ play an important role, see also Proposition 7.4 below. In what follows J will designate a fundamental symmetry of $\mathfrak{K}$ and $(.,.)$ will stand for the Hilbert inner product $[J.,.]$ on $\mathfrak{K}$. The set $\mathfrak{D}[JS]$ is defined to be the domain $\mathfrak{D}(|JS|^{1/2})$ of $|JS|^{1/2}$, computed in the Hilbert space $(\mathfrak{K},(.,.))$. It coincides with the domain $\mathfrak{D}((|JS|+I)^{1/2})$. In [Cu] it is shown that $\mathfrak{D}[JS]$ is independent of the choice of J. From the fact that $|JS|+I$ is boundedly invertible it easily follows that the inner product spaces $(\mathfrak{D}(S),\ ((|JS|+I).,(|JS|+I).))$ and $(\mathfrak{D}[JS],\ ((|JS|+I)^{1/2}.,(|JS|+I)^{1/2}.))$ are Hilbert spaces.

LEMMA 7.3. *Let S be a selfadjoint operator in a Krein space $(\mathfrak{K},[.,.])$ with $\rho(S)\neq\emptyset$. Then the inner product spaces $(\mathfrak{D}(S),[.,.])$ and $(\mathfrak{D}[JS],[.,.])$ are nondegenerate and decomposable, and have unique decomposition majorants.*

Proof. Since $\mathfrak{D}(S)$ and $\mathfrak{D}[JS]$ are dense in $\mathfrak{K}$, it is easily verified that $[.,.]$ does not degenerate on these subspaces. Furthermore, the Hilbert space topologies on $\mathfrak{D}(S)$ and $\mathfrak{D}[JS]$ described above are both majorants of the inner product $[.,.]$ and therefore, the inner product spaces admit Hilbert majorants. By Theorems IV.5.2 and IV.6.4. in [Bo] they are decomposable and have unique decomposition majorants.

The equivalences $(i)\Leftrightarrow(iii)\Leftrightarrow(iv)$ in the following proposition are extensions to definitizable operators of the equivalences $(i)\Leftrightarrow(v)\Leftrightarrow(vii)$ in [Cu] Theorem 2.5.

PROPOSITION 7.4. *Let S be a densely defined, selfadjoint operator in the Krein space $(\mathfrak{K},[.,.])$ with $\rho(S)\neq\emptyset$. Then the following statements are equivalent:*

(i) *There exists a fundamental symmetry J on $\mathfrak{K}$ such that $J\mathfrak{D}[JS]\subset\mathfrak{D}[JS]$.*

(ii) *Each fundamental decomposition of $(\mathfrak{D}[JS],[.,.])$ is uniform.*

(iii) *The decomposition majorant of $(\mathfrak{D}[JS],[.,.])$ is equivalent to the norm topology of $\mathfrak{K}$ restricted to $\mathfrak{D}[JS]$.*

The three statements obtained from $(i),(ii)$ and (iii) by replacing everywhere $\mathfrak{D}[JS]$ by $\mathfrak{D}(S)$ are also equivalent. If S is definitizable, then all six statements are equivalent to:

(iv) *Infinity is not a singular critical point of S.*

Proof. The equivalence of $(i)-(iii)$ and that of the other three statements follow directly from the preceeding lemmas. The implication $(i)\Rightarrow(iv)$ with $\mathfrak{D}(S)$ instead of $\mathfrak{D}[JS]$ follows from [Cu] Theorem 3.2. Now we prove the converse. Denote the spectral function of S by E and let Δ_∞ be such that $\bar{\mathbb{R}}\backslash\Delta_\infty$ is a bounded open interval containing zero and all the finite critical points of S. Let J_0 be a fundamental symmetry on $\mathfrak{K}$ which commutes with $E(\Delta_\infty)$ and put

$$S_\infty = S|_{E(\Delta_\infty)\mathfrak{D}(S)} + J_0|_{E(\bar{\mathbb{R}}\backslash\Delta_\infty)\mathfrak{K}}.$$

Then the operator S_∞ is boundedly invertible and positive in $\mathfrak{K}$, and $\mathfrak{D}(S_\infty)=\mathfrak{D}(S)$. It is easy to see that $\infty\notin c_s(S)$ if and only if $\infty\notin c_s(S_\infty)$, see [Cu] Corollary 3.3. Now the converse implication follows from the implication $(vii)\Rightarrow(i)$ in [Cu] Theorem 2.5 applied to the operator S_∞, cf. [Cu] Lemma 2.4. Finally, Theorem 3.9 in [Cu] yields that $\infty\notin c_s(S_\infty)$ if and only if $\infty\notin c_s(J_0(J_0S_\infty)^{1/2})$. We also have that

$$\mathfrak{D}[JS]=\mathfrak{D}[J_0S]=\mathfrak{D}[J_0S_\infty]=\mathfrak{D}(J_0(J_0S_\infty)^{1/2}),$$

see [Cu] Remark 1.4. The equivalence of (i) and (iv) now follows when we apply what has already been proved to the operator $J_0(J_0S_\infty)^{1/2}$.

Proof of Theorem 7.1. Let $J_{\mathfrak{G}}$ be a fundamental symmetry on the Krein space $(\mathfrak{G},[.,.]_{\mathfrak{G}})$. Denote by $\mathfrak{H}_{\mathfrak{G}}$ the Hilbert space

$$(\mathfrak{D}[J_{\mathfrak{G}}B^{-1}],\ [J_{\mathfrak{G}}|J_{\mathfrak{G}}B^{-1}|^{1/2}.,|J_{\mathfrak{G}}B^{-1}|^{1/2}.]_{\mathfrak{G}}).$$

As shown in [Cu] Remark 1.7, the inner product $[B^{-1}.,.]_{\mathfrak{G}}$ can be extended from $\mathfrak{D}(B^{-1})$ onto $\mathfrak{D}[J_{\mathfrak{G}}B^{-1}]$. We denote this extension also by $[B^{-1}.,.]_{\mathfrak{G}}$. The space $(\mathfrak{D}[J_{\mathfrak{G}}B^{-1}],[B^{-1}.,.]_{\mathfrak{G}})$ is a Pontryagin space. Provided with its Hilbert majorant it coincides with $\mathfrak{H}_{\mathfrak{G}}$ and $B\mathfrak{D}(V)$ is a dense subspace. The same results are valid if we replace $\mathfrak{G}$ by $\mathfrak{F}$, B by A and $\mathfrak{D}(V)$ by $\mathfrak{R}(V)$. From the intertwining relation $AV=VB$ on $\mathfrak{D}(V)$ it follows that for $x=By\in B\mathfrak{D}(V)\subset\mathfrak{D}(V)$ with $y\in\mathfrak{D}(V)$

$$[A^{-1}Vx,Vx]_{\mathfrak{F}}=[Vy,VBy]_{\mathfrak{F}}=[y,By]_{\mathfrak{G}}=[B^{-1}x,x]_{\mathfrak{G}}.$$

We also have that $V(B\mathfrak{D}(V))=A\mathfrak{R}(V)$ and hence, $V|_{B\mathfrak{D}(V)}$ is a weak isomorphism from the Pontryagin space $\mathfrak{D}[J_{\mathfrak{G}}B^{-1}]$ to the Pontryagin space $\mathfrak{D}[J_{\mathfrak{F}}A^{-1}]$. It follows from [IKL] Theorem 6.3 that $V|_{B\mathfrak{D}(V)}$ is a continuous operator and can be extended by continuity to the unitary operator U_0 from the Pontryagin space $\mathfrak{D}[J_{\mathfrak{G}}B^{-1}]$ to the Pontryagin space $\mathfrak{D}[J_{\mathfrak{F}}A^{-1}]$. Consequently, $\kappa=\kappa'$. The inner product $[.,.]_{\mathfrak{G}}$ $([.,.]_{\mathfrak{F}})$ is continuous on the Hilbert space $\mathfrak{H}_{\mathfrak{G}}$ $(\mathfrak{H}_{\mathfrak{F}})$, since the topology on this space is stronger than the norm topology on the Krein space $\mathfrak{G}$ ($\mathfrak{F}$, respectively). As V is a weak isomorphism from $\mathfrak{G}$ to $\mathfrak{F}$,

$$[Vx,Vy]_{\mathfrak{F}}=[x,y]_{\mathfrak{G}},\qquad x,y\in B\mathfrak{D}(V)$$

and hence, by continuity

$$[U_0x,U_0y]_{\mathfrak{F}}=[x,y]_{\mathfrak{G}},\qquad x,y\in\mathfrak{H}_{\mathfrak{G}}.$$

To finish the proof observe that $0\notin c_s(A)$ $(0\notin c_s(B))$ if and only if $\infty\notin c_s(A^{-1})$ $(\infty\notin c_s(B^{-1})$, respectively). Now, the implication *(iv)* $\Rightarrow$ *(iii)* in Proposition 7.4 implies that the completion of $(\mathfrak{D}[J_{\mathfrak{G}}B^{-1}],[.,.]_{\mathfrak{G}})$ with respect to its unique decomposition majorant is exactly the Krein space $\mathfrak{G}$. In other words $\mathfrak{G}$ is the Krein space associated with $\mathfrak{H}_{\mathfrak{G}}$ and $[.,.]_{\mathfrak{G}}$. Analogously, $\mathfrak{F}$ is the Krein space associated with $\mathfrak{H}_{\mathfrak{F}}$ and $[.,.]_{\mathfrak{F}}$. Because $U_0\in\mathbf{L}(\mathfrak{H}_{\mathfrak{G}},\mathfrak{H}_{\mathfrak{F}})$ and is boundedly invertible, Corollary 2.3 implies that U_0 can be extended by continuity to a unitary operator $U\in\mathbf{L}(\mathfrak{G},\mathfrak{F})$. Since $B\mathfrak{D}(V)$ is dense in $\mathfrak{G}$ and since the operator V is closed, it follows that $V=U\in\mathbf{L}(\mathfrak{G},\mathfrak{F})$. This completes the proof of the theorem.

There is an alternative way to deduce the last conclusion in the proof of Theorem 7.1. To show this we first prove the following simple result.

PROPOSITION 7.5. *Let V be a weak isomorphism from the Krein space $(\mathfrak{G},[.,.]_{\mathfrak{G}})$ to the Krein space $(\mathfrak{F},[.,.]_{\mathfrak{F}})$. Suppose that the subspaces $\mathfrak{D}(V)$ and $\mathfrak{R}(V)$ are uniformly decomposable in $\mathfrak{G}$ and $\mathfrak{F}$, respectively. Furthermore, assume that the inner product space $(\mathfrak{R}(V),[.,.]_{\mathfrak{F}})$ has a unique decomposition majorant. Then V is a unitary operator from $\mathfrak{G}$ to $\mathfrak{F}$.*

Proof. Let $\mathfrak{D}(V)=\mathfrak{D}_{+}+\mathfrak{D}_{-}$ be uniform fundamental decomposition of $(\mathfrak{D}(V),[.,.]_{\mathfrak{G}})$. It is easy to see that $\mathfrak{R}(V)=V(\mathfrak{D}_{+})+V(\mathfrak{D}_{-})$ is a fundamental decomposition of $(\mathfrak{R}(V),[.,.]_{\mathfrak{F}})$. Since $\mathfrak{R}(V)$ has a unique decomposition majorant and is uniformly decomposable, this decomposition is uniform. It follows from [An] Chapter 1, that this uniform decomposition, as well as the uniform decomposition $\mathfrak{D}(V)=\mathfrak{D}_{+}+\mathfrak{D}_{-}$, can be extended to the fundamental decompositions of $\mathfrak{F}$ and $\mathfrak{G}$, respectively. Denote the corresponding fundamental symmetries by $J_{\mathfrak{F}}$ and $J_{\mathfrak{G}}$. Then $J_{\mathfrak{F}}V=VJ_{\mathfrak{G}}$ on $\mathfrak{D}(V)$ and this implies the boundedness of V. This completes the proof.

Now notice that Proposition 7.4 and the fact $\infty\notin c_s(B^{-1})$ imply that $\mathfrak{D}[J_{\mathfrak{G}}B^{-1}]=\mathfrak{D}(U_0)$ satisfies the assumptions in Proposition 7.5. Analogously, Proposition 7.4, Lemma 7.3 and the fact that $\infty\notin c_s(A^{-1})$ imply that $\mathfrak{D}[J_{\mathfrak{F}}A^{-1}]=\mathfrak{R}(U_0)$ satisfies the assumptions in Proposition 7.5. Thus, Proposition 7.5 applied to the weak isomorphism U_0 from $\mathfrak{G}$ to $\mathfrak{F}$ yields the last conclusion of the proof of Theorem 7.1.

In [ACG] it is shown that, if $H\in\mathbf{L}(\mathfrak{K})$ and the form $[(I-H^{+}H).,.]$ has κ negative squares, then the form $[(I-HH^{+}).,.]$ also has κ negative squares. This can be used in the last of the following simple consequences of Theorem 7.1. Let $\Delta=(\mathfrak{K},\mathfrak{F},\mathfrak{G};T,F,G,H)$ and $\Delta'=(\mathfrak{K}',\mathfrak{F}',\mathfrak{G}';T',F',G',H')$ be two unitary colligations.

(i) Assume that $\mathfrak{F}=\mathfrak{F}'$, $\mathfrak{G}=\mathfrak{G}'$, and Δ and Δ' are weakly isomorphic. If the operators $I-T^{+}T$ and $I-T'^{+}T'$ have property P_{κ} (for possibly different κ's), then Δ and Δ' are isomorphic (and all κ's are equal).

(ii) Assume that $\mathfrak{K}=\mathfrak{K}'$, and Δ and Δ' coincide weakly. If the operators $I-H^{+}H$, $I-HH^{+}$, $I-H'^{+}H'$ and $I-H'H'^{+}$ have property P_{κ} (for possibly different κ's), then Δ and Δ' coincide (and all κ's are equal).

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