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Integral Equations and Operator Theory

Riesz bases of root vectors of indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions. II

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Abstract. We consider a regular indefinite Sturm-Liouville problem with two self-adjoint boundary conditions affinely dependent on the eigenparameter. We give sufficient conditions under which the root vectors of this Sturm-Liouville problem can be selected to form a Riesz basis of a corresponding weighted Hilbert space.

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1. Introduction

Consider the following eigenvalue problem

$$\begin{aligned} -f''(x) &= \lambda\,(\operatorname{sgn} x) f(x), \quad x \in [-1,1], \\ f'(1) &= \lambda\, f(-1), \\ -f'(-1) &= \lambda\, f(1). \end{aligned}$$

Lengthy but straightforward calculations show the following: there exist an infinite number of real, simple, nonzero eigenvalues which accumulate only at $-\infty$ and $+\infty$; the number 0 is also a simple eigenvalue. Details can be found at the second author's web-site. It is natural to consider this problem in the Hilbert space $L_2(-1,1) \oplus \mathbb{C}^2$. To our knowledge the following related question, which presents interesting mathematical challenges, has not been addressed. Is it possible to select eigenvectors of the given eigenvalue problem to form a Riesz basis of the above Hilbert space? In this article we answer such questions for a wide class of indefinite Sturm-Liouville problems with λ-dependent boundary conditions. In particular, our Theorem 5.2 applies to the above simple example.

We consider a regular indefinite Sturm-Liouville eigenvalue problem of the form

$$-(p\,f')' + q\,f = \lambda\,r\,f \quad \text{on} \quad [-1,1]. \tag{1.1}$$

We assume throughout that the coefficients $1/p, q, r$ in (1.1) are real and integrable over $[-1,1]$, $p(x) > 0$, and $x\,r(x) > 0$ for almost all $x \in [-1,1]$. We impose the following eigenparameter dependent boundary conditions on equation (1.1):

$$\mathsf{M}\mathbf{b}(f) = \lambda\,\mathsf{N}\mathbf{b}(f), \tag{1.2}$$

where M and N are 2×4 matrices and the boundary mapping $\mathbf{b}$ is defined for all f in the domain of (1.1) by

$$\mathbf{b}(f) = \begin{bmatrix} f(-1) & f(1) & (pf')(-1) & (pf')(1) \end{bmatrix}^T.$$

For our opening example

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathsf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We remark that more general boundary conditions have been studied by many authors, recently for example in [3] and [4], but expansion theorems were not considered. Expansion theorems for polynomial boundary conditions and more general operators, but with weight $r = 1$, were given in [11] and [20].

In this article we study the problem (1.1), (1.2) in an operator theoretic setting established in [5]. Under Condition 2.1 below, a definitizable self-adjoint operator A in the Krein space $L_{2,r}(-1,1) \oplus \mathbb{C}^2_\Delta$ (actually A is quasi-uniformly positive as defined in [10]) is associated with the eigenvalue problem (1.1), (1.2). Here Δ is a 2×2 nonsingular Hermitean matrix which is determined by M and N; see Section 2 for details. We remark that the topology of this Krein space is that of the corresponding Hilbert space $L_{2,|r|} \oplus \mathbb{C}^2_{|\Delta|}$. Here, and in the rest of the paper, we abbreviate $L_{2,r}(-1,1)$ to $L_{2,r}$ and $L_{2,|r|}(-1,1)$ to $L_{2,|r|}$. For more details about Krein spaces and their operators see the standard reference [14] and [1] for recent developments.

Our main goal in this paper is to provide sufficient conditions on the coefficients in (1.1), (1.2) under which there is a Riesz basis of the above Hilbert space consisting of the union of bases for all the root subspaces of the above operator A. This will be referred to for the remainder of this section as the *Riesz basis property of* A. We remark that the Riesz basis property of A is equivalent, modulo a finite dimensional subspace, to similarity of A to a self-adjoint operator in a Hilbert space. The latter similarity has been the subject of several recent papers (see for example [15] and [16]) involving Sturm-Liouville expressions on $\mathbb{R}$ without boundary conditions.

Existence of Riesz bases and expansion theorems with a stronger topology, but in a smaller space corresponding to the form domain of the operator A (which in our case is a Pontryagin space), have been considered by many authors; see

[5, 21] and the references there. The results in [5] turned out to be independent of the number and the nature of the boundary conditions and the coefficients p and r. In contrast, the Riesz basis property depends nontrivially on the problem data even for the case when the boundary conditions are λ-independent (corresponding to $\mathsf{N} = 0$ in our notation).

Sufficient conditions on r (near the turning point 0) for the Riesz basis property when $\mathsf{N} = 0$ can be found in [2, 9, 12, 13, 18, 19], for example. That some condition is necessary, even in the case $p = 1$, was shown by Volkmer [22] who proved the existence of an odd r for which the Dirichlet problem (1.1) does not have this property. Recently Parfenov [17] gave a necessary and sufficient condition on an odd weight function r, near its turning point 0, for the Dirichlet problem (1.1) to have the Riesz basis property. In [6] we constructed an odd r for which the Dirichlet problem (1.1) has the Riesz basis property but the anti-periodic problem does not. This example shows that an additional condition on r near the boundary of $[-1, 1]$ (which in some cases behaves as a second turning point, in addition to 0, for (1.1)) is needed for the general case of (1.2). Such conditions are given in [9] for λ-independent boundary conditions and in [7] for exactly one λ-dependent boundary condition (i.e., when N has rank 1).

In this paper we consider the more difficult case of two λ-dependent boundary conditions. The method we use has its origins in the work of Beals [2]. Subsequently it was developed in [8] into a criterion (given below as Theorem 2.2) equivalent to the Riesz basis property of A. This criterion involves a positive homeomorphism W of the Krein space $L_{2,r} \oplus \mathbb{C}^2_\Delta$ with the form domain of A as an invariant subspace. The explicit description of the form domain of A (given in Section 2) depends entirely on the number $k \in \{0, 1, 2\}$ of boundary conditions which do not include derivatives in the λ-terms. We call such boundary conditions *essential*. Note that this differs from the usual terminology for λ-independent conditions. For example, in our terminology $y'(1) = \lambda y(1)$ is an essential boundary condition.

The direct sum structure of the Krein space $L_{2,r} \oplus \mathbb{C}^2_\Delta$ naturally leads us to consider the homeomorphism W as a block operator matrix, the top left entry W_{11} being an operator on $L_{2,r}$. Since it is clear from Section 2 that the functional components of the vectors in the form domain of A are (absolutely) continuous, we see that W_{11} induces a boundary matrix B satisfying

$$\mathsf{B}\begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} = \begin{bmatrix} (W_{11}f)(-1) \\ (W_{11}f)(1) \end{bmatrix}.$$

An important hurdle, with analogues in several of above references, is to solve the inverse problem of finding a suitable W_{11} for a given matrix B. For example, in [7] (see also Section 3 below) such operators W_{11} were constructed with special diagonal B under one-sided Beals type conditions at -1 or 1. In Section 4 we use conditions at -1, at 1, and a condition connecting -1 and 1 to produce W_{11} with an arbitrary prescribed boundary matrix B.

In Sections 5 and 6 we complete the construction of W, thus establishing our sufficient conditions for the Riesz basis property. When there are no essential boundary conditions ($k = 0$), it turns out that the one-sided Beals type condition at 0 suffices; see Theorem 5.1. In other cases, however, we need conditions near the boundary of $[-1, 1]$. Conditions at 0, and at -1 or 1, are sufficient if $k = 2$ and Δ is definite. If Δ is indefinite, then we also need the condition linking -1 and 1. In these cases it suffices to construct W as a block diagonal matrix. This is carried out in Theorem 5.2.

The most difficult case is $k = 1$ which we tackle in Section 6. In this case we need not only off-diagonal blocks for W, but also a perturbation K of W_{11}, where K is an integral operator whose construction is rather delicate. Our final result Theorem 6.1 is as follows. If only one boundary point -1 or 1 appears with λ in the essential boundary condition, then a Beals type condition at that point and at 0 are sufficient. Otherwise we need conditions at both boundary points and at 0, as well as the condition linking -1 and 1.

To conclude this introduction we remark that our conditions simplify drastically if p is even and r is odd, a case which has been studied by several authors [6, 17, 22]. In fact all the conditions that we impose on the boundary are then equivalent; see Example 4.3 and Corollary 6.5.

2. Operators associated with the eigenvalue problem

The maximal operator $S_{\max}$ in $L_{2,r}$ associated with (1.1) is defined by

$$S_{\max} : f \mapsto \ell(f) := \frac{1}{r}\big(-(pf')' + qf\big), \quad f \in \mathcal{D}(S_{\max}),$$

where

$$\mathcal{D}(S_{\max}) = \mathcal{D}_{\max} = \big\{ f \in L_{2,r} : f, pf' \in AC[0,1],\ \ell(f) \in L_{2,r} \big\}.$$

We define the boundary mapping $\mathbf{b}$ by

$$\mathbf{b}(f) = \begin{bmatrix} f(-1) & f(1) & (pf')(-1) & (pf')(1) \end{bmatrix}^T, \quad f \in \mathcal{D}(S_{\max}).$$

and the concomitant matrix Q corresponding to $\mathbf{b}$ by

$$\mathsf{Q} = i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The significance of Q is captured by the following identity

$$\int_{-1}^{1} \big(S_{\max} f \overline{g} - f S_{\max} \overline{g}\big) r = i\, \mathbf{b}(g)^* \mathsf{Q} \mathbf{b}(f), \quad f, g \in \mathcal{D}_{\max}.$$

We note that $\mathsf{Q} = \mathsf{Q}^{-1}$.

Throughout, we shall impose the following nondegeneracy and self-adjointness condition on the boundary data.

Condition 2.1. The boundary matrices M and N in (1.2) satisfy the following:

(1) the 4×4 matrix $\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix}$ is nonsingular,
(2) $\mathsf{MQM}^* = \mathsf{NQN}^* = 0$,
(3) the 2×2 matrix $i\mathsf{MQ}^{-1}\mathsf{N}^*$ is self-adjoint and invertible and we define

$$\Delta := -i\big(\mathsf{MQ}^{-1}\mathsf{N}^*\big)^{-1}.$$

Clearly the boundary value problem (1.1),(1.2) will not change if row reduction is applied to the coefficient matrix

$$\begin{bmatrix} \mathsf{M} & \mathsf{N} \end{bmatrix}. \tag{2.1}$$

In what follows we will assume that the matrix in (2.1) is row reduced to row echelon form (starting the reduction at the bottom right corner). In particular the matrix N has the form

$$\mathsf{N} = \begin{bmatrix} \mathsf{N}_e & 0 \\ \mathsf{N}_1 & \mathsf{N}_n \end{bmatrix}.$$

The matrix 0 in the formula for N is $k \times 2$ with $k \in \{0, 1, 2\}$. The $k \times 2$ matrix N_e and the $(2-k) \times 2$ matrix N_n are of maximal ranks.

There are three possible cases for N in (2.1):

(a) N_n is a 2×2 identity matrix (so $k = 0$),
(b) N_e and N_n are nonsingular 1×2 (row) matrices (so $k = 1$),
(c) N_e is a 2×2 identity matrix (so $k = 2$).

In case (a), both boundary conditions in (1.2) are *non-essential*, that is both rows on the right hand side of (1.2) contain derivatives. In case (b), the boundary condition corresponding to the first row in (1.2) is *essential*, that is no derivatives appear in this row on the right hand side; the second boundary condition in (1.2) is non-essential. In case (c), both boundary conditions in (1.2) are essential. Evidently k is the number of essential boundary conditions.

Next we define a Krein space operator associated with the problem (1.1),(1.2). We consider the linear space $L_{2,r} \oplus \mathbb{C}^2_\Delta$, equipped with the inner product

$$\left[\begin{pmatrix} f \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{v} \end{pmatrix} \right] := \int_{-1}^{1} f\overline{g}r \, + \, \mathbf{v}^*\Delta\mathbf{u}, \quad f, g \in L_{2,r}, \ \mathbf{u}, \mathbf{v} \in \mathbb{C}^2.$$

Then $\big(L_{2,r} \oplus \mathbb{C}^2_\Delta, [\,\cdot\,,\cdot\,]\big)$ is a Krein space. A fundamental symmetry on this Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 \\ 0 & \operatorname{sgn}(\Delta) \end{bmatrix},$$

where 2×2 matrix $\operatorname{sgn}(\Delta)$ and $J_0 : L_{2,r} \to L_{2,r}$ are defined by

$$\operatorname{sgn}(\Delta) = |\Delta|^{-1}\Delta \quad \text{and} \quad (J_0 f)(t) := f(t) \operatorname{sgn}(r(t)), \quad t \in [-1, 1].$$

Then $\langle\,\cdot\,,\,\cdot\,\rangle := [J\,\cdot\,,\,\cdot\,]$ is a positive definite inner product which turns $L_{2,r} \oplus \mathbb{C}^2_\Delta$ into a Hilbert space $\big(L_{2,|r|} \oplus \mathbb{C}^2_{|\Delta|}, \langle\,\cdot\,,\,\cdot\,\rangle\big)$. The topology of $L_{2,r} \oplus \mathbb{C}^2_\Delta$ is defined to be that of $L_{2,|r|}\oplus\mathbb{C}^2_{|\Delta|}$, and a *Riesz basis* of $L_{2,r}\oplus\mathbb{C}^2_{\Delta}$ is defined as a homeomorphic image of an orthonormal basis of $L_{2,|r|}\oplus\mathbb{C}^2_{|\Delta|}$.

We define the operator A in the Krein space $L_{2,r}\oplus\mathbb{C}^2_{\Delta}$ on the domain

$$\mathcal{D}(A)=\left\{\begin{bmatrix} f \\ \mathsf{N}\mathbf{b}(f)\end{bmatrix}\in\mathcal{K}\ :\ f\in\mathcal{D}\big(S_{\max}\big)\right\}$$

by

$$A\begin{bmatrix} f \\ \mathsf{N}\mathbf{b}(f)\end{bmatrix}:=\begin{bmatrix} S_{\max}f \\ \mathsf{M}\mathbf{b}(f)\end{bmatrix},\qquad f\in\mathcal{D}(A).$$

Using [5, Theorems 3.3 and 4.1] we see that this operator is definitizable with discrete spectrum in the Krein space $L_{2,r}\oplus\mathbb{C}^2_{\Delta}$. As in [7, Theorem 2.2], we then obtain the following, which is our basic tool.

Theorem 2.2. *Let $\mathcal{F}(A)$ denote the form domain of A. Then there exists a Riesz basis of $L_{2,r}\oplus\mathbb{C}^2_{\Delta}$ which consists of root vectors of A if and only if there exists a bounded, boundedly invertible, positive operator W in $L_{2,r}\oplus\mathbb{C}^2_{\Delta}$ such that*

$$W\,\mathcal{F}(A)\subset\mathcal{F}(A).$$

In order to apply this result, we need to characterize the form domain $\mathcal{F}(A)$. To this end, let $\mathcal{F}_{\max}$ be the set of all functions f in $L_{2,r}$ which are absolutely continuous on $[-1,1]$ and such that $\int_{-1}^{1}p\,|f'|^2<+\infty$.

By [5, Theorem 4.2], there are three possible cases for the form domain $\mathcal{F}(A)$ of A, corresponding to cases (a), (b) and (c) above.

(a) If $\mathsf{N}_n=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then

$$\mathcal{F}(A)=\left\{\begin{bmatrix} f \\ \mathbf{v}\end{bmatrix}\in\begin{matrix}L_{2,r}\\ \oplus\\ \mathbb{C}^2_{\Delta}\end{matrix}\ :\ f\in\mathcal{F}_{\max},\ \mathbf{v}\in\mathbb{C}^2\right\}. \tag{2.2}$$

(b) If $\mathsf{N}_e=[u\ \ v]$ with $u,v\in\mathbb{C}$ and $|u|^2+|v|^2\neq 0$, then

$$\mathcal{F}(A)=\left\{\begin{bmatrix} f \\ uf(-1)+vf(1) \\ z\end{bmatrix}\in\begin{matrix}L_{2,r}\\ \oplus\\ \mathbb{C}^2_{\Delta}\end{matrix}\ :\ f\in\mathcal{F}_{\max},\ z\in\mathbb{C}\right\}.$$

(c) If $\mathsf{N}_e=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then

$$\mathcal{F}(A)=\left\{\begin{bmatrix} f \\ f(-1) \\ f(1)\end{bmatrix}\in\begin{matrix}L_{2,r}\\ \oplus\\ \mathbb{C}^2_{\Delta}\end{matrix}\ :\ f\in\mathcal{F}_{\max}\right\}. \tag{2.3}$$

To construct an operator W as in Theorem 2.2 we need to impose conditions (to be given in the next two sections) on the coefficients p and r in (1.1). In all cases we need Condition 3.5 in a neighborhood of 0, and in some cases we need one of two Conditions, 3.7 or 3.8, on r in neighborhoods of -1 or 1. These will be discussed in Section 3. In some cases we also need Condition 4.1 connecting the boundary points -1 and 1. This is developed in Section 4.

3. Conditions at 0, -1 and 1

In this section we recall the remaining concepts and results from [7, Sections 3, 4 and 5] which we need in this paper.

A closed interval of non-zero length is said to be a *left half-neighborhood* of its right endpoint and a *right half-neighborhood* of its left endpoint. Let $\imath$ be a closed subinterval of $[-1,1]$. By $\mathcal{F}_{\max}(\imath)$ we denote the set of all functions f in $L_{2,r}(\imath)$ which are absolutely continuous on $\imath$ and such that $\int_\imath p\,|f'|^2 < +\infty$. With this notation we have $\mathcal{F}_{\max} = \mathcal{F}_{\max}[-1,1]$.

Definition 3.1. Let p and r be the coefficients in (1.1). Let $a, b \in [-1,1]$ and let h_a and h_b, respectively, be half-neighborhoods of a and b which are contained in $[-1,1]$. We say that the ordered pair (h_a, h_b) is *smoothly connected* if there exist

(a) positive real numbers ϵ and τ,
(b) non-constant affine functions $\alpha : [0,\epsilon] \to h_a$ and $\beta : [0,\epsilon] \to h_b$,
(c) non-negative real functions ρ and ϖ defined on $[0,\epsilon]$

such that

(i) $\alpha(0) = a$ and $\beta(0) = b$,
(ii) $p \circ \alpha$ and $p \circ \beta$ are locally integrable on the interval $(0,\epsilon]$,
(iii) $\rho \circ \alpha^{-1} \in \mathcal{F}_{\max}\big(\alpha([0,\epsilon])\big)$,
(iv) $1/\tau < \varpi < \tau$ a.e. on $[0,\epsilon]$,
(v) $\rho(t) = \dfrac{|r(\beta(t))|}{|r(\alpha(t))|}$ and $\varpi(t) = \dfrac{p(\beta(t))}{p(\alpha(t))}$ for $t \in (0,\epsilon]$.

The numbers α', β' (the slopes of α, β, respectively) and $\rho(0)$ are called the *parameters* of the smooth connection.

A broad class of examples satisfying this definition can be given via the following one.

Definition 3.2. Let ν and a be real numbers and let h_a be a half-neighborhood of a. Let g be a function defined on h_a. Then g is *of order ν on h_a* if there exists $g_1 \in C^1(h_a)$ such that

$$g(x) = |x-a|^\nu g_1(x) \quad \text{and} \quad g_1(x) \neq 0, \quad x \in h_a.$$

(The absolute value is missing in the corresponding definition in [7]).

Example 3.3. Let $a, b \in \{-1, 0, 1\}$. Let h_a and h_b be half-neighborhoods of a and b, respectively, and contained in $[-1, 1]$. For simplicity assume that $p = 1$. If r in (1.1) has order ν (> -1 to ensure integrability) on both half-neighborhoods h_a and h_b then as noted in [7] the half-neighborhoods h_a and h_b are smoothly connected. Moreover the parameters of the smooth connection are nonzero numbers. We remark that that p can be much more general – see [7, Example 3.4].

Theorem 3.4. *Let $\imath$ and $\jmath$ be closed intervals, $\imath, \jmath \in \big\{[-1, 0], [0, 1]\big\}$. Let a be an endpoint of $\imath$ and let b be an endpoint of $\jmath$. Denote by a_1 and b_1, respectively, the remaining endpoints. Assume that the half-neighborhoods $\imath$ of a and $\jmath$ of b are smoothly connected with parameters α', β' and $\rho(0)$. Then there exists an operator*

$$S : L_{2,|r|}(\imath) \to L_{2,|r|}(\jmath)$$

such that the following hold:

(S-1) $S \in \mathcal{L}\big(L_{2,|r|}(\imath), L_{2,|r|}(\jmath)\big)$, $S^* \in \mathcal{L}\big(L_{2,|r|}(\jmath), L_{2,|r|}(\imath)\big)$;

(S-2) $(Sf)(x) = 0$, $|x - b_1| \le \frac{1}{2}$ *for all* $f \in L_{2,|r|}(\imath)$ *and* $(S^*g)(x) = 0$, $|x - a_1| \le \frac{1}{2}$ *for all* $g \in L_{2,|r|}(\jmath)$;

(S-3) $S\mathcal{F}_{\max}(\imath) \subset \mathcal{F}_{\max}(\jmath)$, $S^*\mathcal{F}_{\max}(\jmath) \subset \mathcal{F}_{\max}(\imath)$;

(S-4) *For all* $f \in \mathcal{F}_{\max}(\imath)$ *and all* $g \in \mathcal{F}_{\max}(\jmath)$ *we have*

$$\lim_{\substack{y \to b \\ y \in \jmath}} (Sf)(y) = |\alpha'| \lim_{\substack{x \to a \\ x \in \imath}} f(x), \quad \lim_{\substack{x \to a \\ x \in \imath}} (S^*g)(x) = |\beta'|\rho(0) \lim_{\substack{y \to b \\ y \in \jmath}} g(y).$$

This is [7, Theorem 3.6].

Condition 3.5 (Condition at 0). Let p and r be coefficients in (1.1). Denote by h_{0-} a generic left and by h_{0+} a generic right half-neighborhood of 0. We assume that at least one of the four ordered pairs of half-neighborhoods

$$(h_{0-}, h_{0-}), \quad (h_{0-}, h_{0+}), \quad (h_{0+}, h_{0-}), \quad (h_{0+}, h_{0+}),$$

is smoothly connected with the connection parameters α_0', β_0' and $\rho_0(0)$ such that $|\alpha_0'| \ne |\beta_0'|\rho_0(0)$.

We note from Example 3.3 that this condition is automatically satisfied if $p = 1$ and r is of order ν on some half-neighborhood of 0.

Theorem 3.6. *Assume that the coefficients p and r satisfy Condition 3.5. Then there exists an operator*

$$W_0 : L_{2,r} \to L_{2,r}$$

such that the following hold:

(a) W_0 *is bounded on* $L_{2,|r|}$;

(b) $J_0W_0 > I$, *in particular* W_0^{-1} *is bounded and* W_0 *is positive on the Krein space* $L_{2,r}$;

(c) $(W_0f)(x) = (J_0f)(x)$, $\frac{1}{2} \le |x| \le 1$, $f \in L_{2,r}$;

(d) $W_0\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$.

This is [7, Theorem 4.2].

Condition 3.7 (Condition at -1). Let p and r be coefficients in (1.1). We assume that a right half neighborhood of -1 is smoothly connected to a right half neighborhood of -1 with the connection parameters $\alpha'_{-1}, \beta'_{-1}$ and $\rho_{-1}(0)$ such that $|\alpha'_{-1}| \neq |\beta'_{-1}|\rho_{-1}(0)$.

Condition 3.8 (Condition at 1). Let p and r be coefficients in (1.1). We assume that a left half-neighborhood of 1 is smoothly connected to a left half-neighborhood of 1 with the connection parameters $\alpha'_{+1}, \beta'_{+1}$ and $\rho_{+1}(0)$ such that $|\alpha'_{+1}| \neq |\beta'_{+1}|\rho_{+1}(0)$.

Again, we note from Example 3.3 that these conditions are automatically satisfied if $p = 1$ and r is of order ν_{-1} and ν_{+1} on some half-neighborhood (in $[-1, 1]$) of -1 and 1, respectively.

The following two propositions appear in [7] as Propositions 5.3 and 5.4, respectively.

Proposition 3.9. *Assume that the coefficients p and r satisfy Condition 3.7. Let b be an arbitrary complex number. Then there exists an operator*

$$W_{-1} : L_{2,r} \to L_{2,r}$$

such that the following hold:

(a) *W_{-1} is bounded on $L_{2,|r|}$;*
(b) *$J_0 W_{-1} > I$, in particular $(W_{-1})^{-1}$ is bounded and W_{-1} is positive on the Krein space $L_{2,r}$;*
(c) *$(W_{-1}f)(x) = (J_0 f)(x), \quad -\frac{1}{2} \le x \le 1, \quad f \in L_{2,r}$;*
(d) *$W_{-1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1, 0] \oplus \mathcal{F}_{\max}[0, 1]$;*
(e) *$(W_{-1}f)(-1) = bf(-1) \quad$ for all $\quad f \in \mathcal{F}_{\max}$.*

Proposition 3.10. *Assume that the coefficients p and r satisfy Condition 3.8. Let b be an arbitrary complex number. Then there exists an operator*

$$W_{+1} : L_{2,r} \to L_{2,r}$$

such that the following hold:

(a) *W_{+1} is bounded on $L_{2,|r|}$;*
(b) *$J_0 W_{+1} > I$, in particular $(W_{+1})^{-1}$ is bounded and W_{+1} is positive on the Krein space $L_{2,r}$;*
(c) *$(W_{+1}f)(x) = (J_0 f)(x), \quad -1 \le x \le \frac{1}{2}, \quad f \in L_{2,r}$;*
(d) *$W_{+1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1, 0] \oplus \mathcal{F}_{\max}[0, 1]$;*
(e) *$(W_{+1}f)(1) = bf(1) \quad$ for all $\quad f \in \mathcal{F}_{\max}$.*

4. Mixed condition at ± 1 and associated operator

In this section we establish analogues of the above results for a new condition involving both endpoints of the interval $[-1, 1]$.

Condition 4.1 (Condition at $-1,1$). Let p and r be the coefficients in (1.1). We assume that at least one of the following three conditions is satisfied.

(A) There are two smooth connections each connecting a right half-neighborhood of -1 to a left half-neighborhood of 1 with the connection parameters α'_{mj}, β'_{mj} and $\rho_{mj}(0)$, $j = 1,2$, such that

$$\begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\rho_{m1}(0) & |\beta'_{m2}|\rho_{m2}(0) \end{vmatrix} \neq 0. \tag{4.1}$$

(B) There are two smooth connections each connecting a left half-neighborhood of 1 to a right half-neighborhood of -1 with the connection parameters $\alpha'_{mj}, \beta'_{mj}$ and $\rho_{mj}(0)$, $j = 1,2$, such that (4.1) holds.

(C) A right half-neighborhood of -1 is smoothly connected to a left half-neighborhood of 1 with the connection parameters $\alpha'_{m1}, \beta'_{m1}$ and $\rho_{m1}(0)$, and a left half-neighborhood of 1 is smoothly connected to a right half-neighborhood of -1 with the connection parameters $\alpha'_{m2}, \beta'_{m2}$ and $\rho_{m2}(0)$, such that

$$\begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}|\rho_{m2}(0) \\ |\beta'_{m1}|\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} \neq 0.$$

Example 4.2. From Example 3.3 it follows that this condition is satisfied if $p = 1$ and r has the same order ν on a right half-neighborhood of -1 and a left half-neighborhood of 1.

Example 4.3. If p is an even function and r is odd, then it turns out that Conditions 3.7, 3.8 and 4.1 are equivalent. The first equivalence is clear. For the second, assume that Condition 3.8 is satisfied. Let α_{+1} and β_{+1} be the corresponding affine functions from Definition 3.1 defined on $[0,\epsilon]$. Now define $\alpha_{m1}(t) = \alpha_{+1}(t), \beta_{m1}(t) = -\beta_{+1}(t)$, $t \in [0,\epsilon)$, so $\rho_{m1} = \rho_{+1}$. Note that p is locally integrable on $[\alpha_{+1}(\epsilon), 1)$ by Definition 3.1 (ii). Then define $\alpha_{m2}(t) = 1-t, \beta_{m2}(t) = -1+t$, $t \in \big[0, 1-\alpha_{+1}(\epsilon)\big)$ and so $\rho_{m2} = 1$. Then Condition 4.1(B) is satisfied since (4.1) takes the form

$$\begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\rho_{m1}(0) & |\beta'_{m2}|\rho_{m2}(0) \end{vmatrix} = \begin{vmatrix} |\alpha'_{+1}| & 1 \\ |\beta'_{+1}|\rho_{+1}(0) & 1 \end{vmatrix}$$

which is nonzero by Condition 3.8. The proof of the converse is similar.

Example 4.4. We call a function $g : [-1,1] \to \mathbb{C}$ *nearly odd* (*nearly even*) if there exists a positive constant $c \neq 1$ such that $g(-x) = -c\,g(x)$ $(g(-x) = c\,g(x))$ for almost all $x \in (0,1]$. We note that if p is a nearly even function and r is nearly odd, both Conditions 3.5 and 4.1 are satisfied. Also, Conditions 3.7 and 3.8 are equivalent. The verification is straightforward.

Example 4.5. Let $p = 1$ and $r(x) = -1$ for $x \in [-1,0)$ and $r(x) = 1-x$ for $x \in [0,1]$. It is not difficult to verify directly that these functions satisfy Conditions 3.5, 3.7 and 3.8, but not Condition 4.1. In addition notice that r is of order 0 in a right half-neighborhood of -1 and of order 1 in a left half-neighborhood of 1.

The proof of the following theorem occupies the remainder of this section.

Theorem 4.6. *Assume that the coefficients p and r satisfy Conditions 3.7, 3.8 and 4.1. Let b_{jk}, $j,k=1,2$, be arbitrary complex numbers. Then there exists an operator*

$$W_{s1} : L_{2,r} \to L_{2,r}$$

such that the following hold:

(a) *W_{s1} is bounded on the Hilbert space $L_{2,|r|}$;*
(b) *$J_0 W_{s1} > I$, in particular W_{s1}^{-1} is bounded and W_{s1} is positive on the Krein space $L_{2,r}$;*
(c) *$(W_{s1}f)(x) = (J_0 f)(x), \quad -\frac{1}{2} \le x \le \frac{1}{2}, \quad f \in L_{2,r}$;*
(d) *$W_{s1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}[-1,0] \oplus \mathcal{F}_{\max}[0,1]$;*
(e)

$$\begin{bmatrix} (W_{s1}f)(-1) \\ (W_{s1}f)(1) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}.$$

Proof. We construct W_{s1} in the form

$$W_{s1} = J_0\big(X_{s1}^* X_{s1} + I\big),$$

where

$$X_{s1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

is a block operator matrix corresponding to the decomposition

$$L_{2,|r|} = L_{2,|r|}(-1,0) \oplus L_{2,|r|}(0,1).$$

We split the proof into three parts. The off-diagonal and diagonal entries of X_{s1} are constructed in the first and second parts, respectively. In the third part we establish the stated properties of W_{s1}.

1. To construct the off-diagonal operators we treat each case (A), (B), (C) of Condition 4.1 separately.

Case (A). By Theorem 3.4 there exist operators

$$S_{mj} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1), \ j = 1,2,$$

which satisfy (S-1)-(S-4) in Theorem 3.4 with $\imath = [-1,0]$, $\jmath = [0,1]$, $a=-1$ and $b=1$. In particular, for $f \in \mathcal{F}_{\max}[-1,0]$ and $j=1,2$,

$$(S_{mj}f)(1) = |\alpha'_{mj}|\, f(-1), \qquad (S_{mj}^* f)(-1) = |\beta'_{mj}|\, \rho_{mj}(0)\, f(1).$$

To simplify the formulas we use the following notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\, \rho_{m1}(0) & |\beta'_{m2}|\, \rho_{m2}(0) \end{vmatrix}.$$

Define

$$X_{21} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1),$$

by

$$X_{21} := b_{21}\,\Upsilon^{-1} \begin{vmatrix} S_{m1} & S_{m2} \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix}.$$

Here and below we write such determinants as abbreviations for corresponding linear combinations of operators. For all $f \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X_{21}f)(1) = b_{21}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}|\,f(-1) & |\alpha'_{m2}|\,f(-1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix} = b_{21}f(-1).$$

Also for all $g \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{21}^{*}g)(-1) = \bar{b}_{21}\,\Upsilon^{-1} \begin{vmatrix} |\beta'_{m1}|\,\rho_{m1}(0)\,g(1) & |\beta'_{m2}|\,\rho_{m2}(0)\,g(1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix} = 0.$$

Now define the opposite off diagonal corner

$$X_{12} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0),$$

by

$$X_{12} := -b_{12}\,\Upsilon^{-1}\,\big(-|\alpha'_{m2}|\;S_{m1}^{*} + |\alpha'_{m1}|\;S_{m2}^{*}\big) = -b_{12}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ S_{m1}^{*} & S_{m2}^{*} \end{vmatrix}.$$

Then for all $f \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{12}f)(-1) = -b_{12}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\,\rho_{m1}(0)\,f(1) & |\beta'_{m2}|\,\rho_{m2}(0)\,f(1) \end{vmatrix} = -b_{12}f(1).$$

Also

$$(X_{12}^{*}f)(1) = -\bar{b}_{12}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\alpha'_{m1}|\,f(-1) & |\alpha'_{m2}|\,f(-1) \end{vmatrix} = 0.$$

Case (B). By Theorem 3.4 there exist operators

$$S_{mj} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0),\ j = 1,2,$$

which satisfy (S-1)-(S-4) in Theorem 3.4 with $\imath = [0,1]$, $\jmath = [-1,0]$, $a = 1$ and $b = -1$. In particular, for all $f \in \mathcal{F}_{\max}[0,1]$ and $j = 1,2$,

$$(S_{mj}f)(-1) = |\alpha'_{mj}|\,f(1), \qquad (S_{mj}^{*}f)(1) = |\beta'_{mj}|\,\rho_{mj}(0)\,f(-1).$$

To simplify the formulas we continue to use the notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix}.$$

Define

$$X_{12} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0),$$

by

$$X_{12} = -b_{12}\,\Upsilon^{-1} \begin{vmatrix} S_{m1} & S_{m2} \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix}.$$

Then for all $f \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{12}f)(-1) = -b_{12}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}|\,f(1) & |\alpha'_{m2}|\,f(1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix} = -b_{12}\,f(1)$$

and for all $g \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X_{12}^{*}g)(1) = -b_{12}\,\Upsilon^{-1} \begin{vmatrix} |\beta'_{m1}|\,\rho_{m1}(0)\,g(-1) & |\beta'_{m2}|\,\rho_{m2}(0)\,g(-1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\beta'_{m2}|\,\rho_{m2}(0) \end{vmatrix} = 0.$$

Now define the opposite off diagonal corner

$$X_{21} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1),$$

by

$$X_{21} = b_{21}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ S_{m1}^{*} & S_{m2}^{*} \end{vmatrix}.$$

Then for all $f \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X_{21}f)(1) = b_{21}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\beta'_{m1}|\,\rho_{m1}(0)\,f(-1) & |\beta'_{m2}|\,\rho_{m2}(0)\,f(-1) \end{vmatrix} = b_{21}\,f(-1)$$

and for all $g \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{21}^{*}g)(-1) = \overline{b}_{21}\,\Upsilon^{-1} \begin{vmatrix} |\alpha'_{m1}| & |\alpha'_{m2}| \\ |\alpha'_{m1}|\,g(1) & |\alpha'_{m2}|\,g(1) \end{vmatrix} = 0.$$

Case (C). By Theorem 3.4 there exists an operator

$$S_{m1} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1)$$

with the properties listed in Case (A) of this proof and there exists an operator

$$S_{m2} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0)$$

with the properties listed in Case (B).

To simplify the formulas in this part of the proof we use the notation

$$\Upsilon := \begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}|\,\rho_{m2}(0) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix}.$$

Define

$$X_{12} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0)$$

by

$$X_{12} = -b_{12}\,\Upsilon^{-1}\begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}|\,\rho_{m2}(0) \\ S^*_{m1} & S_{m2} \end{vmatrix}.$$

Then for all $f \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{12}f)(-1) = -b_{12}\,\Upsilon^{-1}\begin{vmatrix} |\alpha'_{m1}| & |\beta'_{m2}|\,\rho_{m2}(0) \\ |\beta'_{m1}|\,\rho_{m1}(0)\,f(1) & |\alpha'_{m2}|\,f(1) \end{vmatrix} = -b_{12}\,f(1)$$

and for all $g \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X^*_{12}g)(1) = -\bar{b}_{12}\,\Upsilon^{-1}\begin{vmatrix} s_{m1} & \theta_{m2}(0) \\ s_{m1}g(-1) & \theta_{m2}(0)g(-1) \end{vmatrix} = 0.$$

The other off diagonal operator

$$X_{21} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1)$$

is defined as:

$$X_{21} = b_{m21}\,\Upsilon^{-1}\begin{vmatrix} S_{m1} & S^*_{m2} \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix}.$$

Then for all $f \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X_{21}f)(1) = b_{21}\,\Upsilon^{-1}\begin{vmatrix} |\alpha'_{m1}|\,f(-1) & |\beta'_{m2}|\,\rho_{m2}(0)\,f(-1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} = b_{21}\,f(-1)$$

and for all $g \in \mathcal{F}_{\max}[0,1]$ we have

$$(X^*_{21}g)(-1) = \bar{b}_{21}\,\Upsilon^{-1}\begin{vmatrix} |\beta'_{m1}|\,\rho_{m1}(0)\,g(1) & |\alpha'_{m2}|\,g(1) \\ |\beta'_{m1}|\,\rho_{m1}(0) & |\alpha'_{m2}| \end{vmatrix} = 0.$$

We conclude this part of the proof by summarizing that in each of the three cases above we have defined operators

$$X_{12} : L_{2,|r|}(0,1) \to L_{2,|r|}(-1,0) \quad \text{and} \quad X_{21} : L_{2,|r|}(-1,0) \to L_{2,|r|}(0,1)$$

such that

$$\begin{aligned} X_{12}\mathcal{F}_{\max}[0,1] &\subset \mathcal{F}_{\max}[-1,0], & X^*_{12}\mathcal{F}_{\max}[-1,0] &\subset \mathcal{F}_{\max}[1,0], \\ X^*_{21}\mathcal{F}_{\max}[0,1] &\subset \mathcal{F}_{\max}[-1,0], & X_{21}\mathcal{F}_{\max}[-1,0] &\subset \mathcal{F}_{\max}[1,0], \end{aligned}$$

and for all $f \in \mathcal{F}_{\max}[0,1]$ and $g \in \mathcal{F}_{\max}[-1,0]$ we have

$$\begin{aligned} (X_{12}f)(-1) &= -b_{12}\,f(1), & (X^*_{12}g)(1) &= 0, \\ (X^*_{21}f)(-1) &= 0, & (X_{21}g)(1) &= b_{21}\,f(-1). \end{aligned}$$

This completes the construction of the off-diagonal entries of X_{s1}.

2. To construct the diagonal entries we need two self-adjoint operators $P_{1,-}$ and $P_{1,+}$ defined as follows. Let $\phi_1 : [-1,1] \to [0,1]$ be an even function with $\phi_1 \in C^1[-1,1]$ and such that

$$\phi_1(-1) = 1, \qquad \phi_1(x) = 0 \quad \text{for} \quad 0 \le |x| \le 1/2, \qquad \phi_1(1) = 1.$$

We now define

$$P_{1,-} : L_{2,|r|}(-1,0) \to L_{2,|r|}(-1,0) \quad \text{and} \quad P_{1,+} : L_{2,|r|}(0,1) \to L_{2,|r|}(0,1)$$

by

$$(P_{1,-}f)(x) = f(x)\,\phi_1(x), \quad f \in L_{2,|r|}(-1,0), \quad x \in [-1,0], \tag{4.2}$$

and

$$(P_{1,+}f)(x) = f(x)\,\phi_1(x), \quad f \in L_{2,|r|}(0,1), \qquad x \in [0,1]. \tag{4.3}$$

These operators enjoy the following properties:

$$\begin{aligned} (P_{1,-}f)(x) &= 0, \quad f \in L_{2,|r|}(-1,0), \quad -\tfrac{1}{2} \le x \le 0, \\ (P_{1,+}f)(x) &= 0, \quad f \in L_{2,|r|}(0,1), \qquad 0 \le x \le \tfrac{1}{2}, \end{aligned}$$

$$P_{1,-}\mathcal{F}_{\max}[-1,0] \subset \mathcal{F}_{\max}[-1,0], \qquad P_{1,+}\mathcal{F}_{\max}[0,1] \subset \mathcal{F}_{\max}[0,1],$$

and

$$\begin{aligned} (P_{1,-}f)(-1) &= f(-1), \quad f \in \mathcal{F}_{\max}[-1,0], \\ (P_{1,+}f)(1) &= f(1), \qquad f \in \mathcal{F}_{\max}[0,1]. \end{aligned}$$

Now we use Condition 3.7 to construct the operator X_{11}. As in Proposition 3.9, Theorem 3.4 implies that there exists an operator $S_{-1} : L_{2,|r|}(-1,0) \to L_{2,|r|}(-1,0)$ with the properties listed there. In particular for all $f \in \mathcal{F}_{\max}[-1,0]$ we have

$$(S_{-1}f)(-1) = |\alpha'_{-1}|\,f(-1), \qquad (S_{-1}^{*}f)(-1) = |\beta'_{-1}|\,\rho_{-1}(0)\,f(-1).$$

Since $|\alpha'_{-1}| \ne |\beta'_{-1}|\,\rho_{-1}(0)$ we can choose complex numbers γ_1 and γ_2 such that

$$\gamma_1|\alpha'_{-1}| + \gamma_2 = -b_{11} - 1, \qquad \overline{\gamma}_1|\beta'_{-1}|\,\rho_{-1}(0) + \overline{\gamma}_2 = 1.$$

Let $P_{1,-}$ be the operator defined in (4.2). Put

$$X_{11} = \gamma_1\ S_{-1} + \gamma_2\ P_{1,-}.$$

Then for all $f \in \mathcal{F}_{\max}[-1,0]$ we have

$$(X_{11}f)(-1) = (-b_{11} - 1)\,f(-1), \qquad (X_{11}^{*}f)(-1) = f(-1).$$

Note also that

$$X_{11}\mathcal{F}_{\max}[-1,0] \subset \mathcal{F}_{\max}[-1,0] \quad \text{and} \quad X_{11}^{*}\mathcal{F}_{\max}[-1,0] \subset \mathcal{F}_{\max}[-1,0].$$

To construct X_{22} we use Condition 3.8. By Theorem 3.4 there exists a bounded operator

$$S_{+1} : L_{2,|r|}(0,1) \to L_{2,|r|}(0,1)$$

such that

$$S_{+1}\mathcal{F}_{\max}[0,1] \subset \mathcal{F}_{\max}[0,1], \qquad S^*_{+1}\mathcal{F}_{\max}[0,1] \subset \mathcal{F}_{\max}[0,1],$$

and for all $f \in \mathcal{F}_{\max}[0,1]$,

$$(S_{+1}f)(1) = |\alpha'_{+1}|\, f(1), \qquad (S^*_{+1}f)(1) = |\beta'_{+1}|\, \rho_{+1}(0) f(-1).$$

Since $|\alpha'_{+1}| \neq |\beta'_{+1}|\, \rho_{+1}(0)$ we can choose complex numbers δ_1 and δ_2 such that

$$\delta_1 |\alpha'_{+1}| + \delta_2 = -b_{11} - 1, \qquad \overline{\delta}_1 |\beta'_{+1}|\, \rho_{+1}(0) + \overline{\delta}_2 = 1.$$

Let $P_{1,+}$ be the operator defined in (4.3). Put

$$X_{22} = \delta_1\, S_{+1} + \delta_2\, P_{1,+}.$$

Then for all $f \in \mathcal{F}_{\max}[0,1]$ we have

$$(X_{22}f)(1) = (b_{22} - 1)\, f(1) \quad \text{and} \quad (X^*_{22}f)(1) = f(1).$$

Note also that

$$X_{22}\mathcal{F}_{\max}[0,1] \subset \mathcal{F}_{\max}[0,1] \quad \text{and} \quad X^*_{22}\mathcal{F}_{\max}[0,1] \subset \mathcal{F}_{\max}[0,1].$$

3. Now we formally define $W_{s1} := J_0(X^*_{s1}X_{s1} + I)$ where

$$X_{s1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

To complete the proof, we verify the properties of W_{s1} stated in the theorem. Indeed, (a) and (b) are immediate, and since $(X_{ij}f)(x) = 0$ whenever $-\frac{1}{2} \leq x \leq \frac{1}{2}$, (c) follows. Moreover, each of the operators X_{ij} maps $\mathcal{F}_{\max}[-1,0]$ or $\mathcal{F}_{\max}[0,1]$ to $\mathcal{F}_{\max}[-1,0]$ or $\mathcal{F}_{\max}[0,1]$ according to its position in the matrix, so (d) holds.

Finally, we check the effect of the individual components at the boundary points -1 and 1. Evidently

$$X_{s1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}, \qquad X^*_{s1}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}.$$

Moreover for $f, g \in \mathcal{F}_{\max}$ we have

$$\begin{bmatrix} (X_{s1}f)(-1) \\ (X_{s1}f)(1) \end{bmatrix} = \begin{bmatrix} (X_{11}f)(-1) + (X_{12}f)(-1) \\ (X_{21}f)(1) + (X_{22}f)(1) \end{bmatrix} = \begin{bmatrix} (-b_{11} - 1)f(-1) - b_{12}f(1) \\ b_{21}f(-1) + (b_{22} - 1)f(1) \end{bmatrix}$$

and

$$\begin{bmatrix} (X^*_{s1}g)(-1) \\ (X^*_{s1}g)(1) \end{bmatrix} = \begin{bmatrix} (X^*_{11}g)(-1) + (X^*_{21}g)(-1) \\ (X^*_{12}g)(1) + (X^*_{22}g)(1) \end{bmatrix} = \begin{bmatrix} g(-1) + 0 \\ 0 + g(1) \end{bmatrix}.$$

Substituting $g = X_{s1}f \in \mathcal{F}_{\max}$, we get

$$\begin{bmatrix} (X^*_{s1}X_{s1}f)(-1) \\ (X^*_{s1}X_{s1}f)(1) \end{bmatrix} = \begin{bmatrix} (-b_{11} - 1)f(-1) - b_{12}f(1) \\ b_{21}f(-1) + (b_{22} - 1)f(1) \end{bmatrix}.$$

With $Y_{s1} = X_{s1}^* X_{s1} + I$ we have

$$\begin{bmatrix} (Y_{s1}f)(-1) \\ (Y_{s1}f)(1) \end{bmatrix} = \begin{bmatrix} -b_{11}f(-1) - b_{12}f(1) \\ b_{21}f(-1) + b_{22}f(1) \end{bmatrix} = \begin{bmatrix} -b_{11} & -b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix},$$

which proves (e) since $W_{s1} = J_0 Y_{s1}$. □

Remark 4.7. Notice that the operators W_{-1} and W_{+1} from Propositions 3.9 and 3.10 satisfy

$$\begin{bmatrix} (W_{-1}f)(-1) \\ (W_{-1}f)(1) \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (W_{+1}f)(-1) \\ (W_{+1}f)(1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix},$$

respectively, with arbitrary $b \in \mathbb{C}$. A stronger conclusion is contained in Theorem 4.6 (e) under stronger assumptions.

5. Two essential or two non-essential boundary conditions

The first theorem of this section deals with the case of two non-essential boundary conditions.

Theorem 5.1. *Assume that the following two conditions are satisfied.*

(a) $\mathsf{N}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) *The coefficients p and r satisfy Condition* 3.5.

Then there is a basis for each root subspace of A, so that the union of all these bases is a Riesz basis of $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

Proof. By (2.2), the form domain of A is given as

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ \mathbf{v} \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}^2_{\Delta} \end{matrix} : f \in \mathcal{F}_{\max}, \, \mathbf{v} \in \mathbb{C}^2 \right\}.$$

Recalling W_0 from Theorem 3.6, we easily see that the operator

$$W = \begin{bmatrix} W_0 & 0 \\ 0 & \mathsf{\Delta}^{-1} \end{bmatrix} : \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}^2_{\Delta} \end{matrix} \to \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}^2_{\Delta} \end{matrix}.$$

is bounded, boundedly invertible and positive in the Krein space $L_{2,r} \oplus \mathbb{C}^2_{\Delta}$. A simple verification shows that $W\,\mathcal{F}(A) \subset \mathcal{F}(A)$ so the theorem follows from Theorem 2.2. □

We now consider the case of two essential conditions.

Theorem 5.2. *Assume that the following three conditions are satisfied.*

(a) $\mathsf{N}_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) *The coefficients p and r satisfy Condition* 3.5.

(c) *One of the following holds:*
   (i) $\Delta > 0$ *and the coefficients* p *and* r *satisfy Condition* 3.7.
   (ii) $\Delta < 0$ *and the coefficients* p *and* r *satisfy Condition* 3.8.
   (iii) *the coefficients* p *and* r *satisfy Conditions* 3.7, 3.8 *and* 4.1.

Then there is a basis for each root subspace of A*, so that the union of all these bases is a Riesz basis of* $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

Proof. Define the following two Krein spaces:

$$\mathcal{K}_0 := L_{2,r}\left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \qquad \mathcal{K}_1 := L_{2,r}(-1, -\tfrac{1}{2})[\dot{+}]L_{2,r}(\tfrac{1}{2}, 1).$$

Extending functions in $\mathcal{K}_0$ and $\mathcal{K}_1$ by zero, we consider the spaces $\mathcal{K}_0$ and $\mathcal{K}_1$ as subspaces of $L_{2,r}$. Then

$$L_{2,r} = \mathcal{K}_0[\dot{+}]\mathcal{K}_1.$$

As in the previous proof our goal is to construct $W : L_{2,r} \oplus \mathbb{C}^2_{\Delta} \to L_{2,r} \oplus \mathbb{C}^2_{\Delta}$. The first step is to define $W_{01} : L_{2,r} \to L_{2,r}$. We proceed by considering each case in (c) separately.

(i) Let W_0 be the operator constructed in Theorem 3.6 and let W_{-1} be the operator constructed in Proposition 3.9 with $b = 1$. Property (c) in Theorem 3.6 and Proposition 3.9 imply that $\mathcal{K}_0$ and $\mathcal{K}_1$ are invariant under W_0 and W_{-1}. Since we chose $b = 1$, we have $(W_{-1}f)(-1) = f(-1)$ and $(W_{-1}f)(1) = f(1)$. Define

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{-1}|_{\mathcal{K}_1}. \tag{5.1}$$

Since W_0 and W_{-1} are bounded, boundedly invertible and positive in the Krein space $L_{2,r}$, so is the the operator W_{01}. Also, $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$ and

$$\begin{bmatrix} (W_{01}f)(-1) \\ (W_{01}f)(1) \end{bmatrix} = \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}. \tag{5.2}$$

(ii) Instead of W_{-1} in (i), we use the operator W_{+1} constructed in Proposition 3.10 with $b = -1$. Redefining the operator W_{01} as

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{+1}|_{\mathcal{K}_1}. \tag{5.3}$$

we see that it is again bounded, boundedly invertible, and positive in the Krein space $L_{2,r}$, $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$ and (since we use $b = -1$)

$$\begin{bmatrix} (W_{01}f)(-1) \\ (W_{01}f)(1) \end{bmatrix} = - \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}. \tag{5.4}$$

(iii) This time we replace W_{-1} from (i) by W_{s1} from Theorem 4.6, so we define the operator

$$W_{01} := W_0|_{\mathcal{K}_0}[\dot{+}]W_{s1}|_{\mathcal{K}_1}, \tag{5.5}$$

which is again bounded, boundedly invertible and positive in the Krein space $L_{2,r}$. Also, $W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}$ and

$$\begin{bmatrix}(W_{s1}f)(-1)\\(W_{s1}f)(1)\end{bmatrix} = \Delta^{-1}\begin{bmatrix}f(-1)\\f(1)\end{bmatrix}. \tag{5.6}$$

Finally we define $W : L_{2,r} \oplus \mathbb{C}^2_\Delta \to L_{2,r} \oplus \mathbb{C}^2_\Delta$ by

$$W = \begin{bmatrix}W_{01} & 0\\0 & I\end{bmatrix} \tag{5.7}$$

in case (c)(i),

$$W = \begin{bmatrix}W_{01} & 0\\0 & -I\end{bmatrix} \tag{5.8}$$

in case (c)(ii), and

$$W = \begin{bmatrix}W_{01} & 0\\0 & \Delta^{-1}\end{bmatrix} \tag{5.9}$$

in case (c)(iii).

By (2.3), the form domain of A is

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ f(-1) \\ f(1) \end{bmatrix} \in \begin{matrix} L_{2,r} \\ \oplus \\ \mathbb{C}^2_\Delta \end{matrix} \;:\; f \in \mathcal{F}_{\max} \right\}.$$

A straightforward verification shows that in each case (5.7), (5.8), and (5.9), W is a bounded, boundedly invertible, positive operator in the Krein space $L_{2,r} \oplus \mathbb{C}^2_\Delta$. Moreover $W\,\mathcal{F}(A) \subset \mathcal{F}(A)$ via (5.2), (5.4) or (5.6). Now the theorem follows from Theorem 2.2. □

Example 5.3. Consider the eigenvalue problem

$$\begin{aligned}-f'' &= \lambda\, r\, f\\ f'(1) &= \lambda f(-1)\\ -f'(-1) &= \lambda f(1),\end{aligned}$$

where $r(x) = \operatorname{sgn} x, x \in [-1,1]$, as in our example in the Introduction. Then clearly $\mathsf{N}_e = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, giving (a) in Theorem 5.2 and (b) follows from the note after Condition 3.5. Moreover, an easy computation gives $\Delta = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, which is indefinite. Condition (c) now follows from Examples 4.2 and 4.3, so Theorem 5.2 applies.

On the other hand, if instead we take r as in Example 4.5, then as we have seen, Condition 4.1 fails and hence so does (c)(iii) in Theorem 5.2. Therefore Theorem 5.2 gives no conclusion about a Riesz basis for this amended case.

6. One essential and one non-essential boundary condition

The main result of this section is the following theorem. Its proof will occupy the most of the section and then we will proceed to some examples.

Theorem 6.1. *Assume that the following three conditions are satisfied.*

(a) $\mathsf{N} = \begin{bmatrix} u & v & 0 & 0 \\ * & * & * & 1 \end{bmatrix}$ *or* $\mathsf{N} = \begin{bmatrix} u & v & 0 & 0 \\ * & * & 1 & 0 \end{bmatrix}$, *where* $|u|^2 + |v|^2 > 0$ *and the asterisks stand for arbitrary complex numbers.*

(b) *The coefficients* p *and* r *satisfy Condition* 3.5.

(c) *One of the following holds.*

  (i) $u = 1, v = 0$ *and the coefficients* p *and* r *satisfy Condition* 3.7.

  (ii) $u = 0, v = 1$ *and the coefficients* p *and* r *satisfy Condition* 3.8.

  (iii) $uv \neq 0$ *and the coefficients* p *and* r *satisfy Conditions* 3.7, 3.8 *and* 4.1.

Then there is a basis for each root subspace of A, *so that the union of all these bases is a Riesz basis of* $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

Proof. It follows from (a) that the form domain of A is

$$\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ uf(-1) + vf(1) \\ z \end{bmatrix} \in \begin{array}{c} L_{2,r} \\ \oplus \\ \mathbb{C}^2_{\Delta} \end{array} : f \in \mathcal{F}_{\max},\ z \in \mathbb{C} \right\}. \tag{6.1}$$

It is no restriction if we scale the first boundary condition so that

$$|u|^2 + |v|^2 = 1. \tag{6.2}$$

As in the previous proofs we shall construct $W : L_{2,r} \oplus \mathbb{C}^2_{\Delta} \to L_{2,r} \oplus \mathbb{C}^2_{\Delta}$ in blocks. We divide the proof into three parts and two lemmas.

1. First we define a bounded operator $W_{01} : L_{2,r} \to L_{2,r}$ such that

$$J_0 W_{01} > I, \tag{6.3}$$

$$W_{01}\mathcal{F}_{\max} \subset \mathcal{F}_{\max}, \tag{6.4}$$

$$u(W_{01}f)(-1) + v(W_{01}f)(1) = 0, \qquad f \in \mathcal{F}_{\max}. \tag{6.5}$$

We distinguish the three cases in (c) above.

(i) As in the proof of Theorem 5.2(i), we define W_{01} by (5.1), but now using $b = 0$ instead of $b = 1$. Then W_{01} is a bounded operator in the Krein space $L_{2,r}$, and it satisfies (6.4) and $(W_{01}f)(-1) = 0$, $(W_{01}f)(1) = f(1)$ and hence (6.5). Inequality (6.3) follows from (5.1), Theorem 3.6(b) and Proposition 3.9(b).

(ii) This time we define W_{01} by (5.3), but now using $b = 0$ instead of $b = -1$. Then W_{01} is a bounded operator in the Krein space $L_{2,r}$, it satisfies (6.4) and $(W_{01}f)(-1) = -f(-1)$, $(W_{01}f)(1) = 0$ and hence (6.5). In this case inequality (6.3) follows from (5.3), Theorem 3.6(b) and Proposition 3.10(b).

(iii) We now define W_{01} as in the proof of Theorem 5.2(iii), but instead of using Δ^{-1} in 5.6 we use the zero 2×2 matrix $\mathsf{0}$. Then W_{01} is a bounded operator in the

Krein space $L_{2,r}$, it satisfies (6.4) and $(W_{01}f)(-1) = 0$, $(W_{01}f)(1) = 0$ and hence (6.5). Inequality (6.3) follows from (5.5), Theorem 3.6 (b) and Theorem 4.6 (b).

2. Next we define an integral operator K which will be a perturbation of W_{01}.
2.1. We start by writing the inverse of the matrix Δ in the form

$$\Delta^{-1} = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \overline{\eta}_{12} & \eta_{22} \end{bmatrix},$$

and setting $\eta := \max\{|\eta_{11}|, |\eta_{12}|\} > 0$, with $\delta_2 \geq \delta_1 > 0$ as the eigenvalues of $|\Delta|$. We also define three positive constants

$$\alpha := \frac{\delta_2}{1 + 2\|r\|_1 \delta_2 \eta^2},$$

$$c := \frac{\alpha}{2\delta_2} \sqrt{\frac{\delta_1}{2}}, \tag{6.6}$$

$$\kappa := \frac{2\delta_2 \eta^2 \|r\|_1}{1 + 2\|r\|_1 \delta_2 \eta^2} = 2\,\alpha\eta^2 \|r\|_1. \tag{6.7}$$

Notice that

$$1 - \kappa = \frac{1}{1 + 2\|r\|_1 \delta_2 \eta^2} = \frac{\alpha}{\delta_2}. \tag{6.8}$$

2.2. Since r is integrable over $[-1, 1]$, we there exists $\gamma \in [0, 1)$ such that

$$-\int_{-1}^{-\gamma} r + \int_{\gamma}^{1} r \leq \left(\frac{c}{\alpha\eta}\right)^2. \tag{6.9}$$

Noting that $p^{-1/2} \in L_2(0, 1) \subset L_1(0, 1)$ we can define

$$\phi(x) = \int_0^x p^{-1/2} \chi_{[\gamma,1]}, \quad x \in [0, 1].$$

Extending ϕ as an even function over $[-1, 1]$ we see that $\phi \in \mathcal{F}_{\max}$. Since $\phi(1)$ is a positive real number, we define $\psi = \phi/\phi(1)$. Clearly $\psi : [-1, 1] \to [0, 1]$ is an even function in $\mathcal{F}_{\max}$ such that

$$\psi(-1) = 1, \quad \psi(0) = 0, \quad \psi(1) = 1, \tag{6.10}$$

and, by (6.9),

$$\|\psi\|_{2,|r|} \leq \frac{c}{\alpha\eta}. \tag{6.11}$$

2.3. Define

$$\psi_j(x) = \begin{cases} \alpha\,\eta_{1j}\,\overline{u}\;\psi(x), & x \in [-1, 0), \\ \alpha\,\eta_{1j}\,\overline{v}\;\psi(x), & x \in [0, 1]. \end{cases} \tag{6.12}$$

Since $\psi \in \mathcal{F}_{\max}$ and $\psi(0) = 0$, the functions ψ_1 and ψ_2 belong to $\mathcal{F}_{\max}$. Set

$$\omega(x) := \eta_{11}\,\overline{\psi_1(x)} + \eta_{12}\,\overline{\psi_2(x)}, \quad x \in [-1, 1],$$

and define $k : [-1,1] \times [-1,1] \to \mathbb{C}$ by

$$k(x,t) = \begin{cases} u\,\overline{\omega(x)} & \text{if} \quad t \le -|x|, \\ \overline{v}\,\omega(t) & \text{if} \quad x > |t|, \\ v\,\overline{\omega(x)} & \text{if} \quad t \ge |x|, \\ \overline{u}\,\omega(t) & \text{if} \quad x < -|t|. \end{cases} \tag{6.13}$$

By the definitions of ψ_1, ψ_2 and ω, since ψ is a nonnegative even function, for all $x \in [0,1]$ we have

$$\overline{u}\,\omega(-x),\ \overline{v}\,\omega(x) \in \mathbb{R}, \quad \text{and} \quad \overline{v}\,\omega(-x) = u\,\overline{\omega(x)}. \tag{6.14}$$

Since ω is continuous, it follows from (6.14) and (6.13) that k is a continuous function. Moreover, by (6.2) and (6.12),

$$|\omega(t)| < \eta\,\eta\alpha + \eta\,\eta\alpha = 2\eta^2\alpha.$$

Therefore (6.7) shows that

$$|k(x,t)| \le 2\eta^2\alpha = \frac{\kappa}{\|r\|_1}. \tag{6.15}$$

The first of our two lemmas is as follows.

Lemma 6.2. *Let $K : L_{2,r} \to L_{2,r}$ be the integral operator defined by*

$$(Kf)(x) := \int_{-1}^{1} k(x,t)\, f(t)\, r(t)\, dt, \qquad f \in L_{2,r}.$$

Then

(I) *The operator K is bounded and self-adjoint on $L_{2,r}$ and $\|K\|_{2,|r|} \le \kappa$.*
(II) *The range of K is contained in $\mathcal{F}_{\max}$.*

Proof. (I) We first note that for f in $L_{2,r}$ the function fr is integrable on $(-1,1)$. In fact

$$\begin{aligned} \int_{-1}^{1} |fr| &= \int_{-1}^{1} |r|^{1/2}\big(|f||r|^{1/2}\big) \\ &\le \left(\int_{-1}^{1} |r|\right)^{1/2} \left(\int_{-1}^{1} |f|^2|r|\right)^{1/2} = \|r\|_1^{1/2}\|f\|_{2,r}. \end{aligned} \tag{6.16}$$

For $f \in L_{2,|r|}$ we calculate

$$\begin{aligned} \|Kf\|_{2,|r|}^2 &\le \int_{-1}^{1}\int_{-1}^{1} |k(x,t)|\,|f(t)|\,|r(t)|dt \int_{-1}^{1} |k(x,s)|\,|f(s)|\,|r(s)|\,ds\,|r(x)|dx \\ &\le \frac{\kappa^2}{\|r\|_1^2} \int_{-1}^{1} \left(\int_{-1}^{1} |f|\,|r|\right)^2 |r(x)|dx \le \kappa\|f\|_{2,|r|}^2, \end{aligned}$$

by virtue of (6.15) and (6.16). Thus $\|K\|_{2,|r|} \le \kappa$, so K is bounded, and self-adjointness follows from (6.13) since $k(x,t) = \overline{k(t,x)}$, $x,t \in [-1,1]$.

(II) Let $f \in L_{2,r}$. By definition, for $-1 \leq x < 0$,

$$(Kf)(x) \;=\; u\,\overline{\omega(x)}\,\int_{-1}^{x}(fr)(t)dt \,+\, \overline{u}\int_{x}^{-x}\big(\omega fr\big)(t)dt \,+\, v\,\overline{\omega(x)}\,\int_{-x}^{1}(fr)(t)dt$$

and, for $0 < x \leq 1$,

$$(Kf)(x) \;=\; u\,\overline{\omega(x)}\,\int_{-1}^{-x}(fr)(t)dt \,+\, \overline{v}\int_{-x}^{x}\big(\omega fr\big)(t)dt \,+\, v\,\overline{\omega(x)}\,\int_{x}^{1}(fr)(t)dt.$$

The function fr is integrable on $(-1,1)$ by (6.16). Since $\omega \in \mathcal{F}_{\max}$ the function ωfr is also integrable on $(-1,1)$. Moreover,

$$\lim_{x\uparrow 0}(Kf)(x) = \lim_{x\downarrow 0}(Kf)(x) = (Kf)(0) = 0.$$

Therefore for each $f \in L_{2,|r|}$ the function Kf is absolutely continuous on $[-1,1]$. For almost all $x \in [-1,0)$, we have

$$(Kf)'(x) = u\,\overline{\omega}'(x)\,\int_{-1}^{x}(fr)(t)dt + v\,\overline{\omega}'(x)\,\int_{-x}^{1}(fr)(t)dt$$
$$+\,u\,\overline{\omega(x)}\,(fr)(x) - \overline{u}\,\omega(x)\,\big(fr\big)(x) - \overline{u}\,\omega(-x)\,\big(fr\big)(-x) + v\,\overline{\omega(x)}\,(fr)(-x),$$

and, for almost all $x \in (0,1]$,

$$(Kf)'(x) = u\overline{\omega}'(x)\int_{-1}^{-x}(fr)(t)dt + v\overline{\omega}'(x)\int_{x}^{1}(fr)(t)dt$$
$$-\,u\,\overline{\omega(x)}\,(fr)(-x) + \overline{v}\,\omega(-x)\,\big(fr\big)(-x) + \overline{v}\,\omega(x)\,\big(fr\big)(x) - v\,\overline{\omega(x)}\,(fr)(x).$$

By (6.14) the terms not involving integrals in the above two equations cancel in pairs. Thus $Kf \in \mathcal{F}_{\max}$ for all $f \in L_{2,|r|}$ since $\overline{\omega} \in \mathcal{F}_{\max}$. This completes the proof of the lemma. $\square$

3. We create off-diagonal blocks for W by means of the operator $Z : \mathbb{C}^2_{\boldsymbol{\Delta}} \to L_{2,r}$ which we define by

$$Z\mathbf{a} := a_1\,\psi_1 + a_2\,\psi_2, \qquad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2.$$

The adjoint $Z^{[*]} : L_{2,r} \to \mathbb{C}^2_{\boldsymbol{\Delta}}$ of Z is given by

$$Z^{[*]}f = \Delta^{-1}\begin{bmatrix} [f,\psi_1] \\ [f,\psi_2] \end{bmatrix}, \quad f \in L_{2,r}.$$

Equalities (6.2), (6.11) and (6.12) yield $\|\psi_1\|_{2,|r|} \leq c$ and $\|\psi_2\|_{2,|r|} \leq c$. Therefore

$$\int_{-1}^{1}\big|Z\mathbf{a}\big|^2|r| \leq 2\big(|a_1|^2\|\psi_1\|^2_{2,|r|} + |a_2|^2\|\psi_2\|^2_{2,|r|}\big)$$
$$\leq 2c^2\mathbf{a}^*\mathbf{a} \leq 2\frac{c^2}{\delta_1}\mathbf{a}^*|\Delta|\mathbf{a}.$$

Consequently, by (6.6),

$$\|Z\| = \|Z^{[*]}\| \le c\sqrt{\frac{2}{\delta_1}} = \frac{\alpha}{2\delta_2}. \tag{6.17}$$

The second lemma we need is as follows.

Lemma 6.3. *Let the operator* $W : L_{2,r} \oplus \mathbb{C}^2_{\Delta} \to L_{2,r} \oplus \mathbb{C}^2_{\Delta}$ *be defined by*

$$W := \begin{bmatrix} W_{01} + K & Z \\ Z^{[*]} & \alpha\,\Delta^{-1} \end{bmatrix}.$$

Then

(I) W *is bounded and uniformly positive on* $L_{2,r} \oplus \mathbb{C}^2_{\Delta}$.
(II) $W\,\mathcal{F}(A) \subset \mathcal{F}(A)$.

Proof. (I) The operator W is bounded since each of its components is bounded. To prove that W is uniformly positive, we shall show that the operator JW is uniformly positive in the Hilbert space $L_{2,|r|} \oplus \mathbb{C}^2_{|\Delta|}$. From Lemma 6.2, $\|K\| = \|JK\| \le \kappa$ and

$$\left\| \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \right\| = \left\| J \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \right\| \le \frac{\alpha}{2\delta_2}$$

follows from (6.17). Thus

$$\begin{aligned}
\left\langle JW \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} J_0 W_{01} & 0 \\ 0 & \alpha|\Delta|^{-1} \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\
&\quad + \left\langle \begin{bmatrix} J_0 K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle + \left\langle J \begin{bmatrix} 0 & Z \\ Z^{[*]} & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\
&= \langle J_0 W_{01} f, f\rangle + \alpha\,\mathbf{a}^*\mathbf{a} + \langle J_0 K f, f\rangle + \left\langle J \begin{bmatrix} 0 & Z \\ Z^{\langle * \rangle} & 0 \end{bmatrix} \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle \\
&\ge \langle f, f\rangle + \frac{\alpha}{\delta_2}\,\mathbf{a}^*|\Delta|\mathbf{a} - \kappa\langle f, f\rangle - \frac{\alpha}{2\delta_2}\left(\langle f, f\rangle + \mathbf{a}^*|\Delta|\mathbf{a}\right) \\
&\ge \left(1 - \kappa - \frac{\alpha}{2\delta_2}\right)\langle f, f\rangle + \left(\frac{\alpha}{\delta_2} - \frac{\alpha}{2\delta_2}\right)\mathbf{a}^*|\Delta|\mathbf{a} \\
&= \left(\frac{\alpha}{\delta_2} - \frac{\alpha}{2\delta_2}\right)\langle f, f\rangle + \frac{\alpha}{2\delta_2}\,\mathbf{a}^*|\Delta|\mathbf{a} \qquad \text{(by (6.8))} \\
&= \frac{\alpha}{2\delta_2}\left(\langle f, f\rangle + \mathbf{a}^*|\Delta|\mathbf{a}\right) \\
&= \frac{\alpha}{2\delta_2}\left\langle \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix}, \begin{bmatrix} f \\ \mathbf{a} \end{bmatrix} \right\rangle,
\end{aligned}$$

as required.

(II) We start with the identity

$$u(Kf)(-1) + v(Kf)(1) = \eta_{11}\,[f, \psi_1] + \eta_{12}\,[f, \psi_2], \quad f \in L_{2,|r|}, \tag{6.18}$$

which follows from the calculation

$$\begin{aligned}
u(Kf)(-1) &+ v(Kf)(1) \\
&= u\int_{-1}^{1} k(-1,t)\, f(t)\, r(t)\, dt + v\int_{-1}^{1} k(1,t)\, f(t)\, r(t)\, dt \\
&= u\int_{-1}^{1} \overline{u}\big(\eta_{11}\overline{\psi}_1(t) + \eta_{12}\overline{\psi}_2(t)\big)\, f(t)\, r(t)\, dt \\
&\qquad + v\int_{-1}^{1} \overline{v}\big(\eta_{11}\overline{\psi}_1(t) + \eta_{12}\overline{\psi}_2(t)\big)\, f(t)\, r(t)\, dt \\
&= |u|^2\eta_{11}[f,\psi_1] + |u|^2\eta_{12}[f,\psi_2] + |v|^2\eta_{11}[f,\psi_1] + |v|^2\eta_{12}[f,\psi_2] \\
&= \eta_{11}[f,\psi_1] + \eta_{12}[f,\psi_2].
\end{aligned}$$

By (6.1), the general element of $\mathcal{F}(A)$ takes the form

$$\begin{bmatrix} f \\ uf(-1)+vf(1) \\ z \end{bmatrix}$$

where $f \in \mathcal{F}_{\max}$ and $z \in \mathbb{C}$. Applying W to this vector we obtain

$$w := \begin{bmatrix} g \\ \eta_{11}\,[f,\psi_1] + \eta_{12}\,[f,\psi_2] + \alpha\,\eta_{11}\,\big(uf(-1)+vf(1)\big) + \alpha\,\eta_{12}\,z \\ * \end{bmatrix},$$

where

$$g := W_{01}f + Kf + \big(uf(-1)+vf(1)\big)\,\psi_1 + z\,\psi_2 \in \mathcal{F}_{\max}$$

by (6.4) and Lemma 6.2. Thus to prove that $w \in \mathcal{F}(A)$, it is enough to show that

$$u\,g(-1)+v\,g(1) = \eta_{11}\,[f,\psi_1]+\eta_{12}\,[f,\psi_2]+\alpha\,\eta_{11}\,\big(uf(-1)+vf(1)\big)+\alpha\,\eta_{12}\,z. \quad (6.19)$$

To this end we calculate

$$\begin{aligned}
u\,g(-1) &= u\,\big((W_{01}f)(-1) + (Kf)(-1) + \big(uf(-1)+vf(1)\big)\,\psi_1(-1) + z\,\psi_2(-1)\big) \\
&= u\,(W_{01}f)(-1) + u\,(Kf)(-1) + \alpha\,|u|^2\eta_{11}\,\big(uf(-1)+vf(1)\big) + \alpha\,|u|^2\,\eta_{12}\,z
\end{aligned}$$

from (6.10) and (6.12). Similarly

$$\begin{aligned}
v\,g(1) &= v\,\big((W_{01}f)(1) + (Kf)(1) + \big(uf(-1)+vf(1)\big)\,\psi_1(1) + z\,\psi_2(1)\big) \\
&= v\,(W_{01}f)(1) + v\,(Kf)(1) + \alpha\,|v|^2\,\eta_{11}\,\big(uf(-1)+vf(1)\big) + \alpha\,|v|^2\,\eta_{12}\,z.
\end{aligned}$$

Adding and using (6.5), (6.18) and (6.2), we obtain (6.19). This completes the proof of the lemma. □

The theorem now follows from Theorem 2.2 and Lemma 6.3. □

We now specialize Theorems 5.1, 5.2 and 6.1 to some of our earlier examples. First we consider Example 3.3 (cf. Example 4.2).

Corollary 6.4. *Assume that* $p = 1$ *and* r *is of order* $\nu_0 > -1$ *on a half-neighborhood of* 0*, and of order* $\nu_1 > -1$ *on both a right half-neighborhood of* -1 *and a left half-neighborhood of* 1*. Then there is a basis for each root subspace of* A*, so that the union of all these bases is a Riesz basis of* $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

Now we consider Examples 4.3 and 4.4.

Corollary 6.5. *Assume that* p *is even,* r *is odd and that Condition* 3.5 *holds. If* $k = 0$ *or Condition* 3.7 *holds, then there is a basis for each root subspace of* A*, so that the union of all these bases is a Riesz basis of* $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

As a simple illustration of this corollary we could consider the eigenvalue problem stated in Example 5.3 but with r odd and of order ν_0 at 0 and ν_1 at 1 (and hence of order ν_1 at -1, since r is odd).

Corollary 6.6. *Assume that* p *is nearly even and* r *is nearly odd. If* $k = 0$ *or Condition* 3.7 *holds, then there is a basis for each root subspace of* A*, so that the union of all these bases is a Riesz basis of* $L_{2,|r|} \oplus \mathbb{C}_{|\Delta|}$.

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