Perturbations of Roots under Linear Transformations of Polynomials

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Circles in a Circle

1923 Oil on canvas

38 7/8 x 37 5/8 inches (98.7 x 95.6 cm)

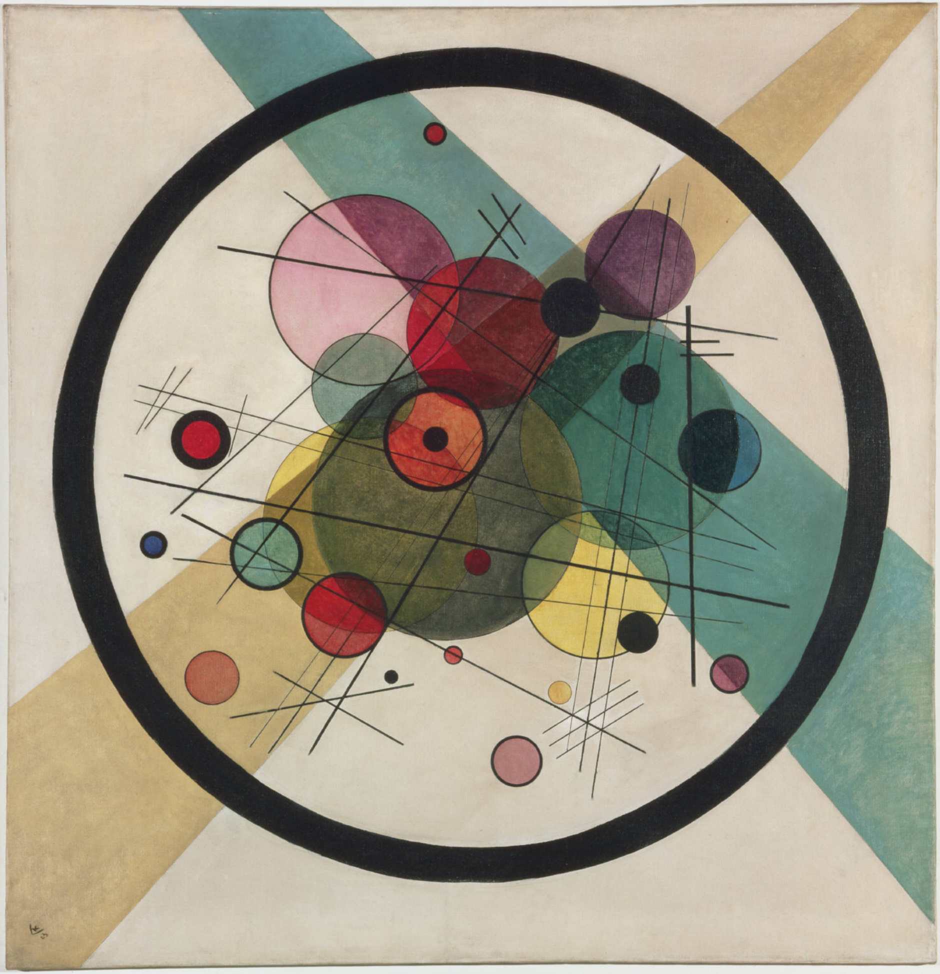
Wassily Kandinsky

Russian, worked in Germany and France

lived 1866 - 1944

Philadelphia Museum of Art:

The Louise and Walter Arensberg Collection



- Q. I. Rahman, G. Schmeisser:
  Analytic theory of polynomials.
  Oxford University Press, 2002.

- T. Sheil-Small:
  Complex polynomials.
  Cambridge University Press, 2002.

- M. Marden:
  Geometry of polynomials. Second edition,
  American Mathematical Society, 1966.

This is joint research with Vania Mascioni.

- On the location of critical points of polynomials.
  Proc. Amer. Math. Soc. 131 (2003), 253–264.

- A contraction of the Lucas polygon.
  Proc. Amer. Math. Soc. 132 (2004), 2973–2981.

- Roots and polynomials as homeomorphic spaces.
  Expositiones Mathematicae 24 (2006), 81–95.

◦ Results of this talk are from a paper
  accepted in Constructive Approximation.

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$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

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$$\mathcal{P}_n = \left\{ \text{ all polynomials of degree } \leq n \right\}$$

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Vladimir Tulovsky: On perturbations of roots of polynomials.
J. Analyse Math. 54 (1990), 77–89.

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$Z(p) = Z(Tp)$ for all non-constant $p \in \mathcal{P}_n$

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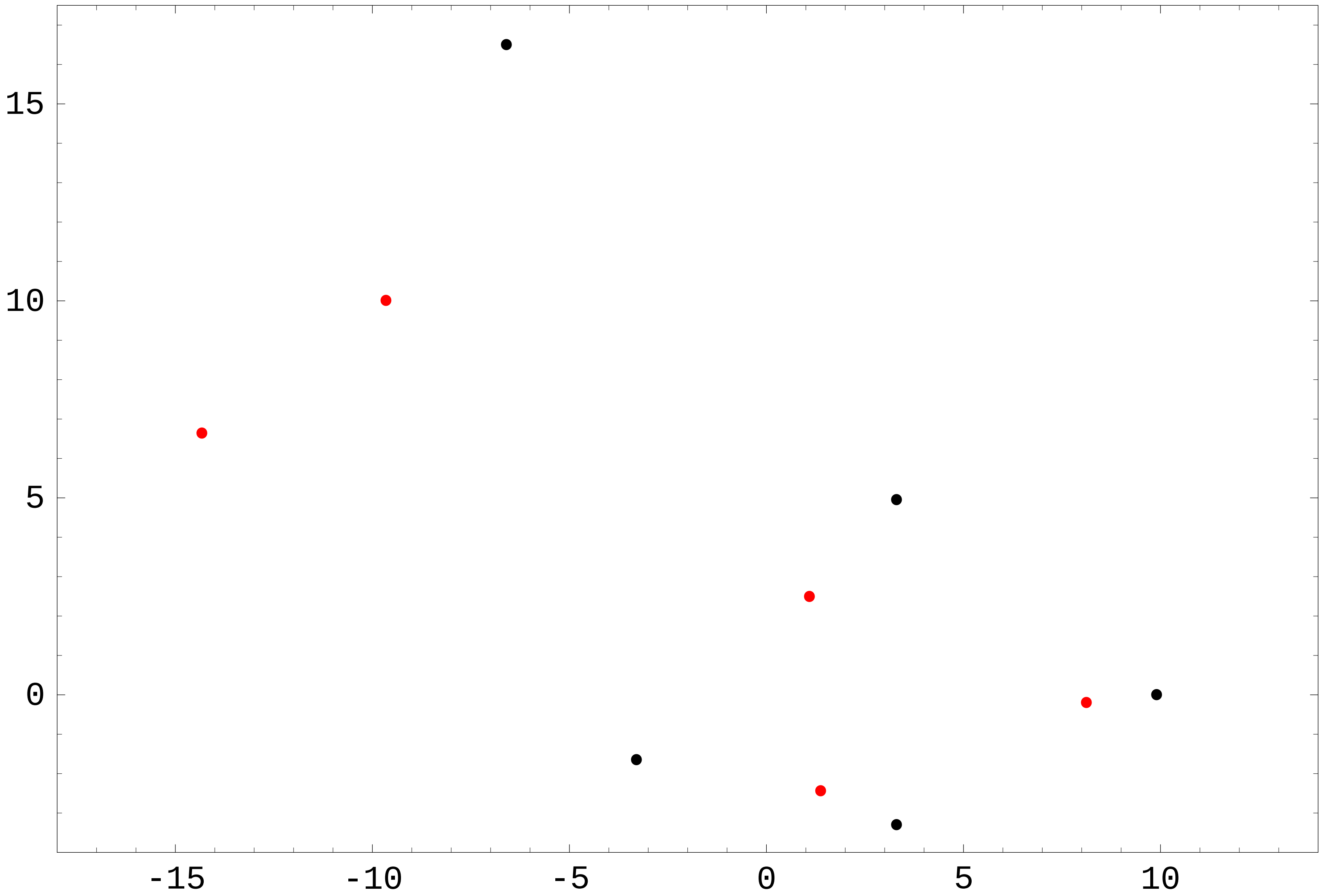
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Let $T \in \mathcal{L}(\mathcal{P}_n)$ and $\mathbb{D}(r) = \left\{ z \in \mathbb{C} : |z| \leq r \right\}$.

$\exists\ C_T > 0$ such that

$\left(Z(p) + \mathbb{D}(C_T) \right) \cap Z(Tp) \neq \emptyset$ for all non-constant $p \in \mathcal{P}_n$

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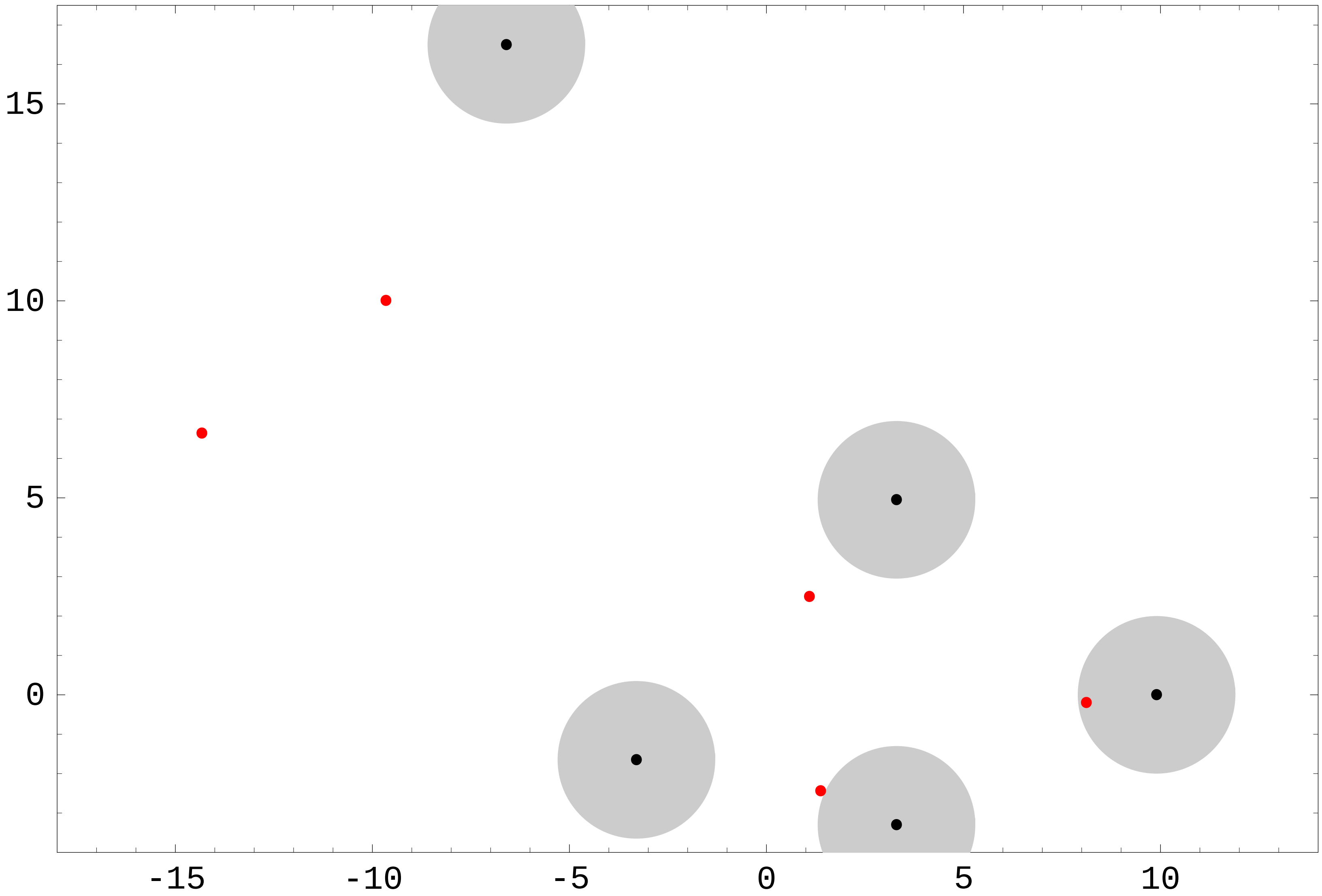

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Then $\boxed{Z(Tp) \subset Z(p) + \mathbb{D}(K_T)}$ for all non-constant $p \in \mathcal{P}_n$.

Let U and V be finite subsets of $\mathbb{C}$.

The Hausdorff distance is defined by:

$$d_H(U,V) = \min\Big\{r > 0 : V \subset U + \mathbb{D}(r),\ U \subset V + \mathbb{D}(r)\Big\}.$$

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Then

$$d_H\Big(Z(p), Z(Tp)\Big) \leq \max\Big\{K_T, K_{T^{-1}}\Big\}$$

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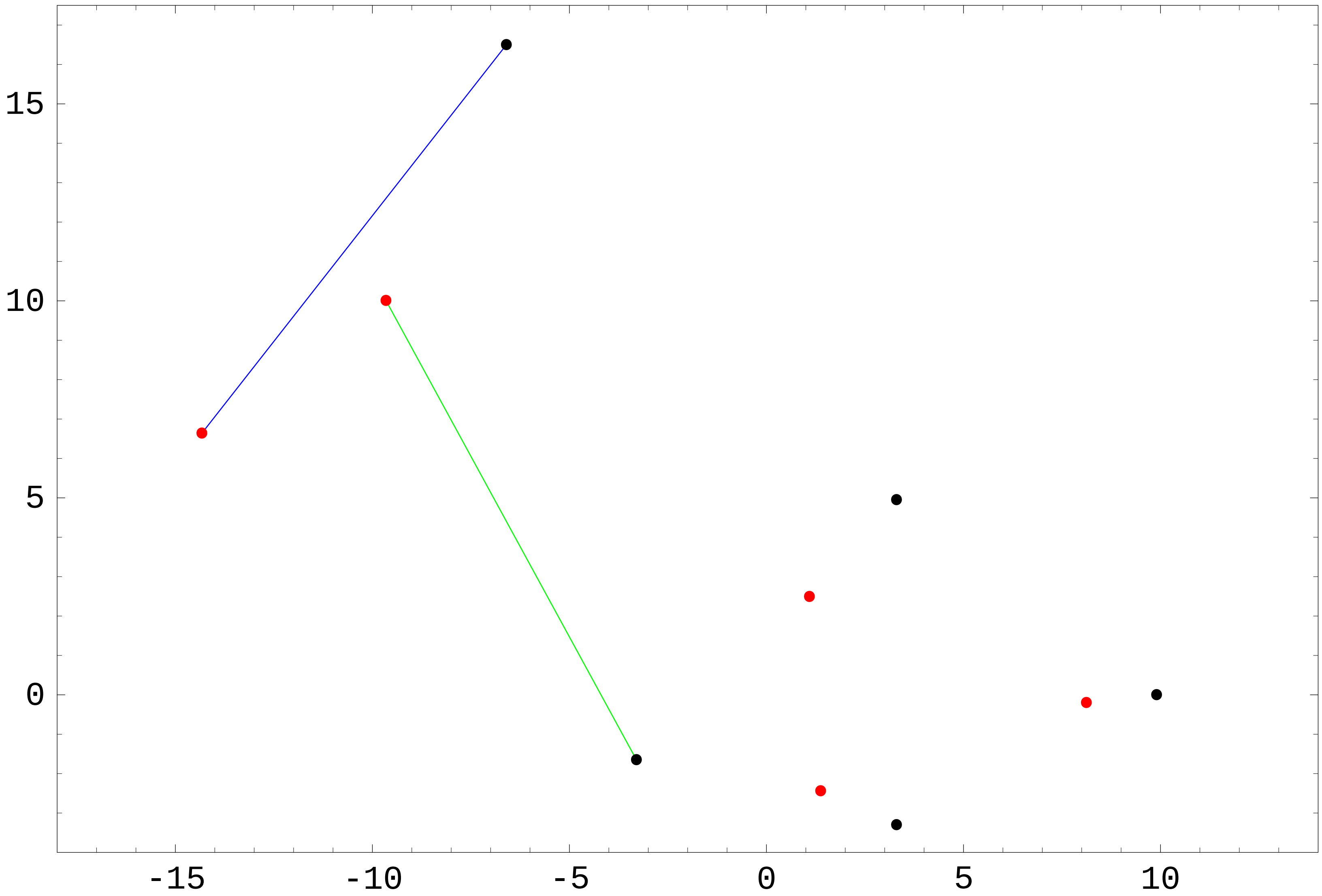
The Fréchet distance is defined by:

$$d_F(U, V) := \min_{\sigma \in \Pi_m} \max_{1 \le k \le m} \left| u_k - v_{\sigma(k)} \right|.$$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Let $\quad T = \alpha_0 I + \alpha_1 D + \cdots + \alpha_n D^n, \quad \alpha_0 \neq 0.$

Let $\gamma_1, \ldots, \gamma_n$ be the roots of

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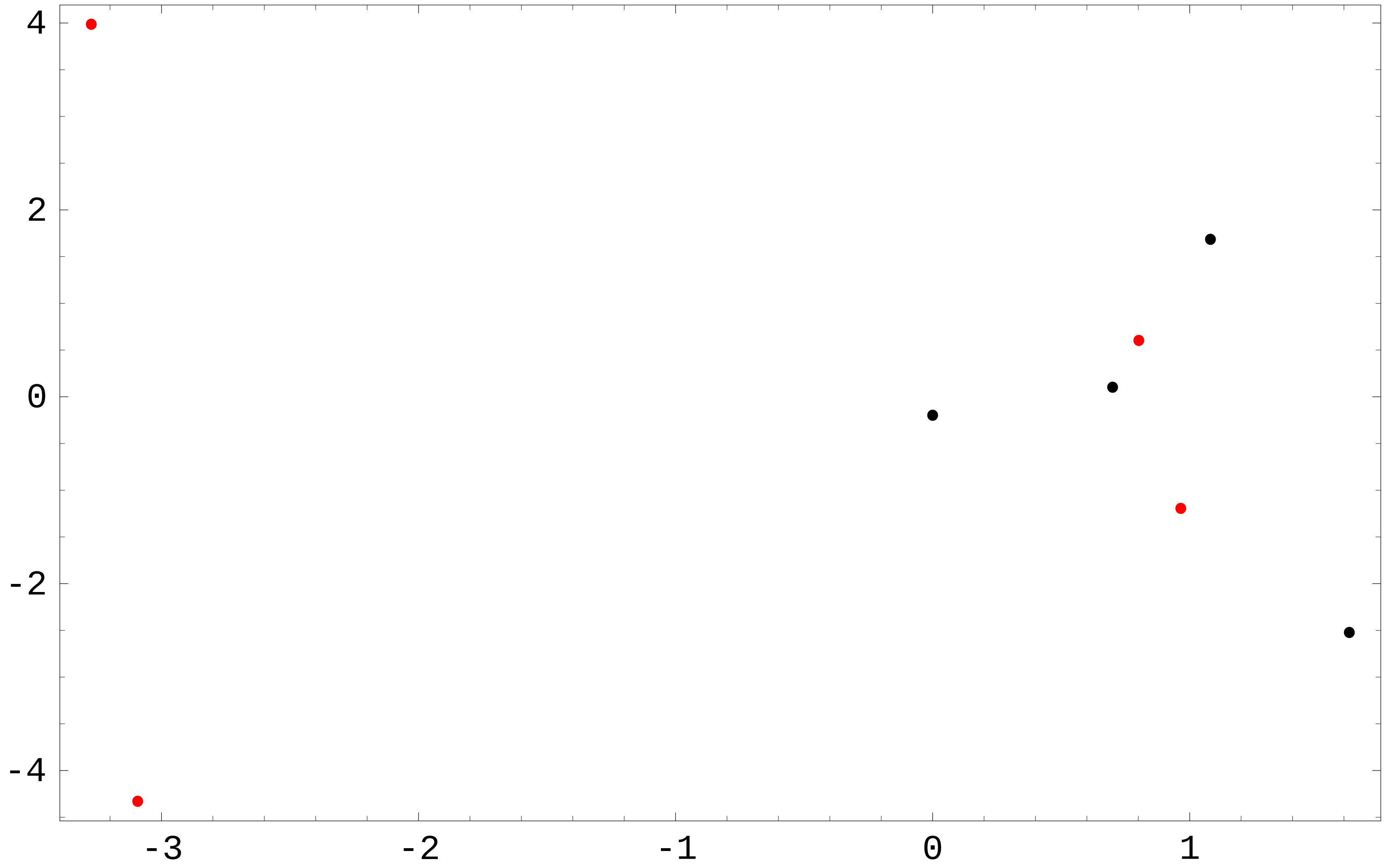
Then

$$d_F\Big(Z(p), Z(Tp)\Big) \leq n^2\Big(|\gamma_1| + \cdots + |\gamma_n|\Big)$$

for all non-constant $p \in \mathcal{P}_n$.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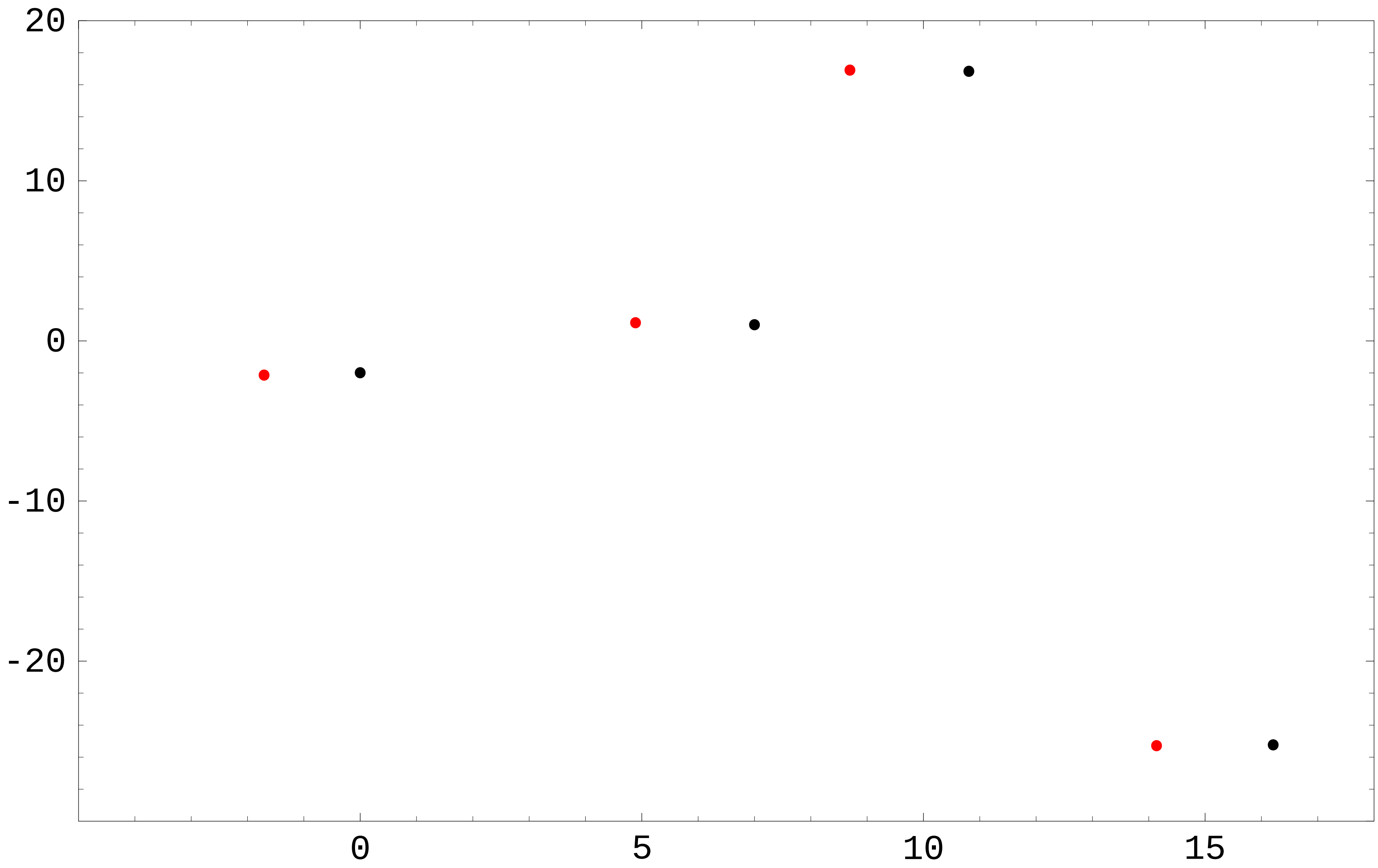

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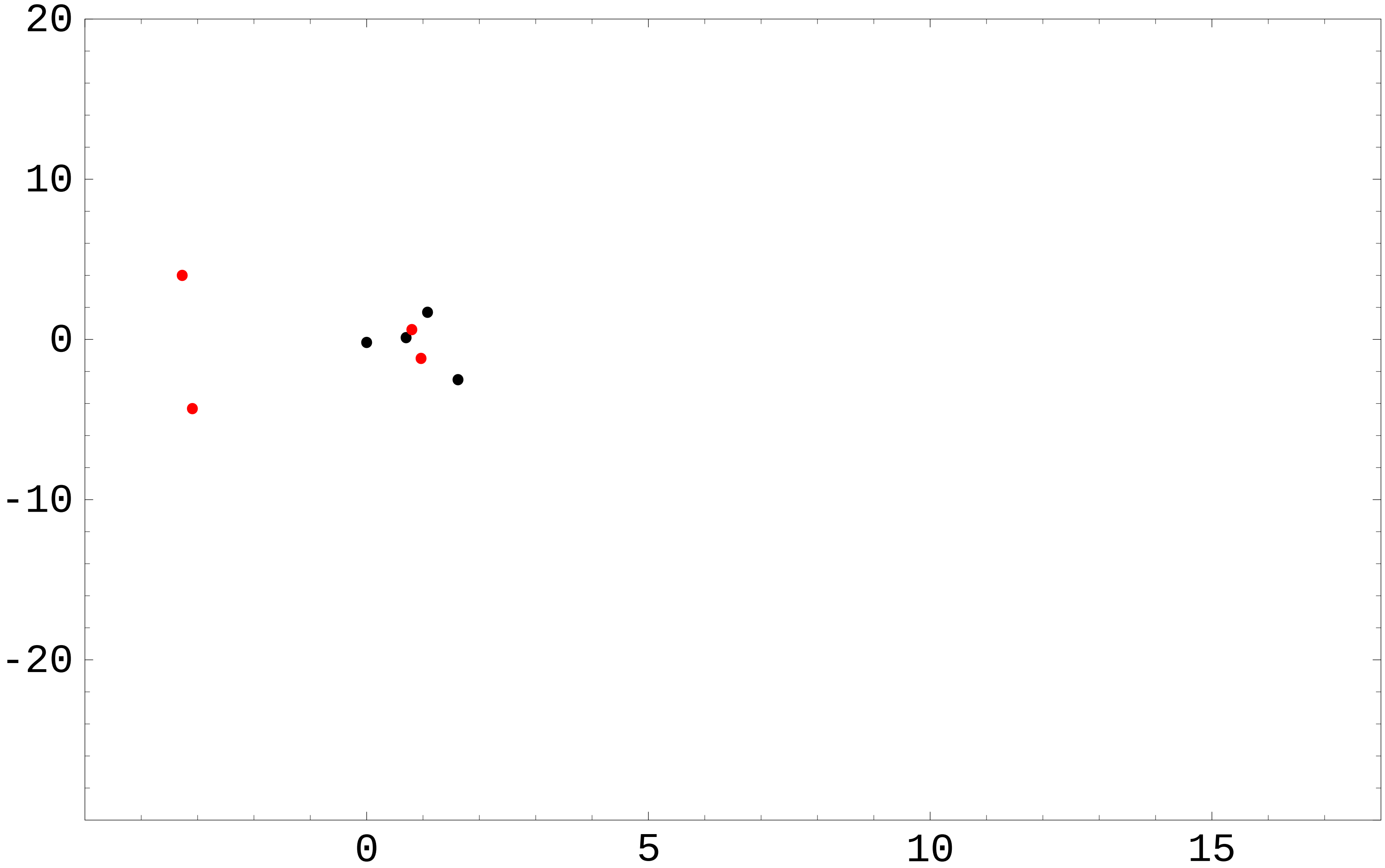
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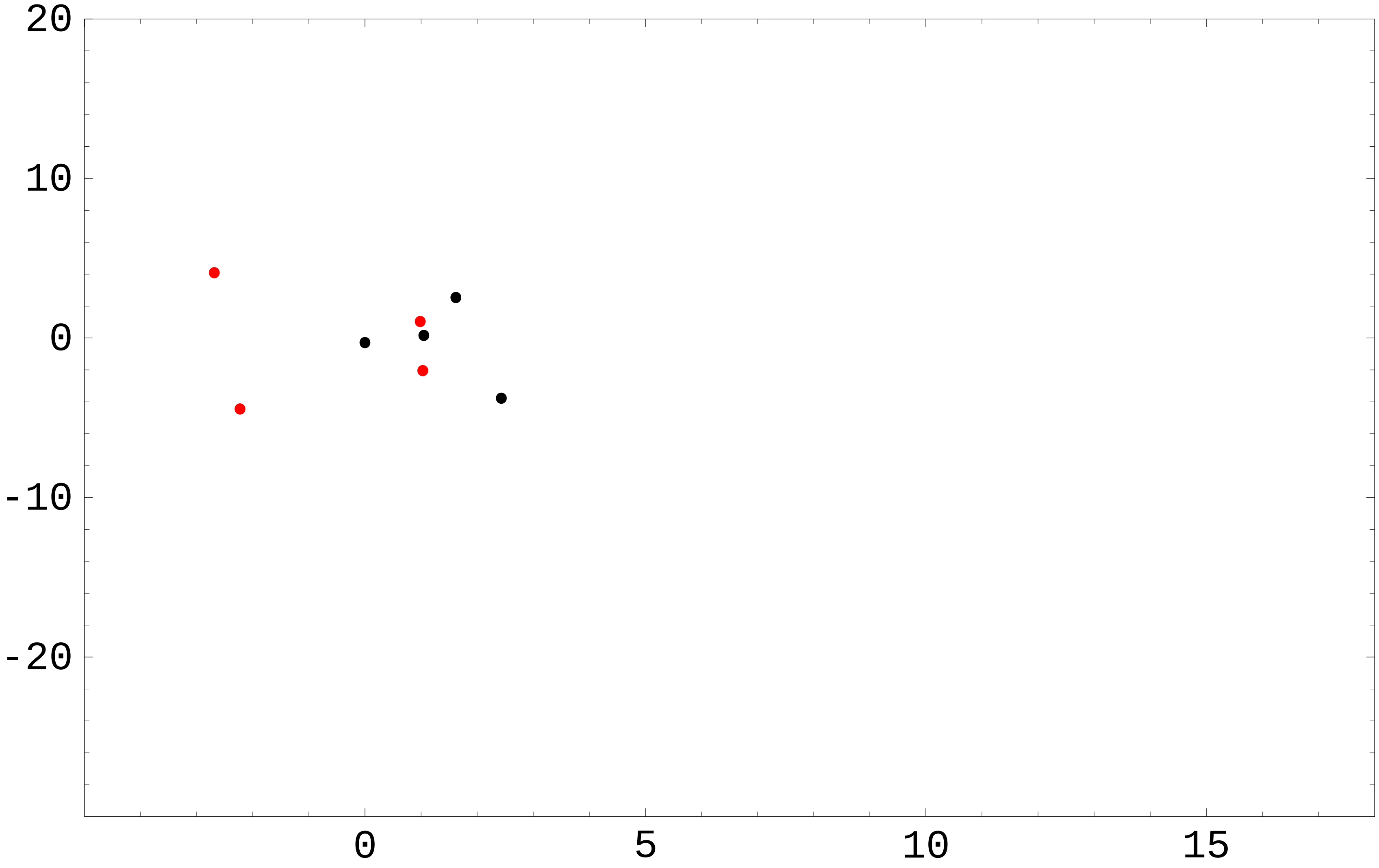
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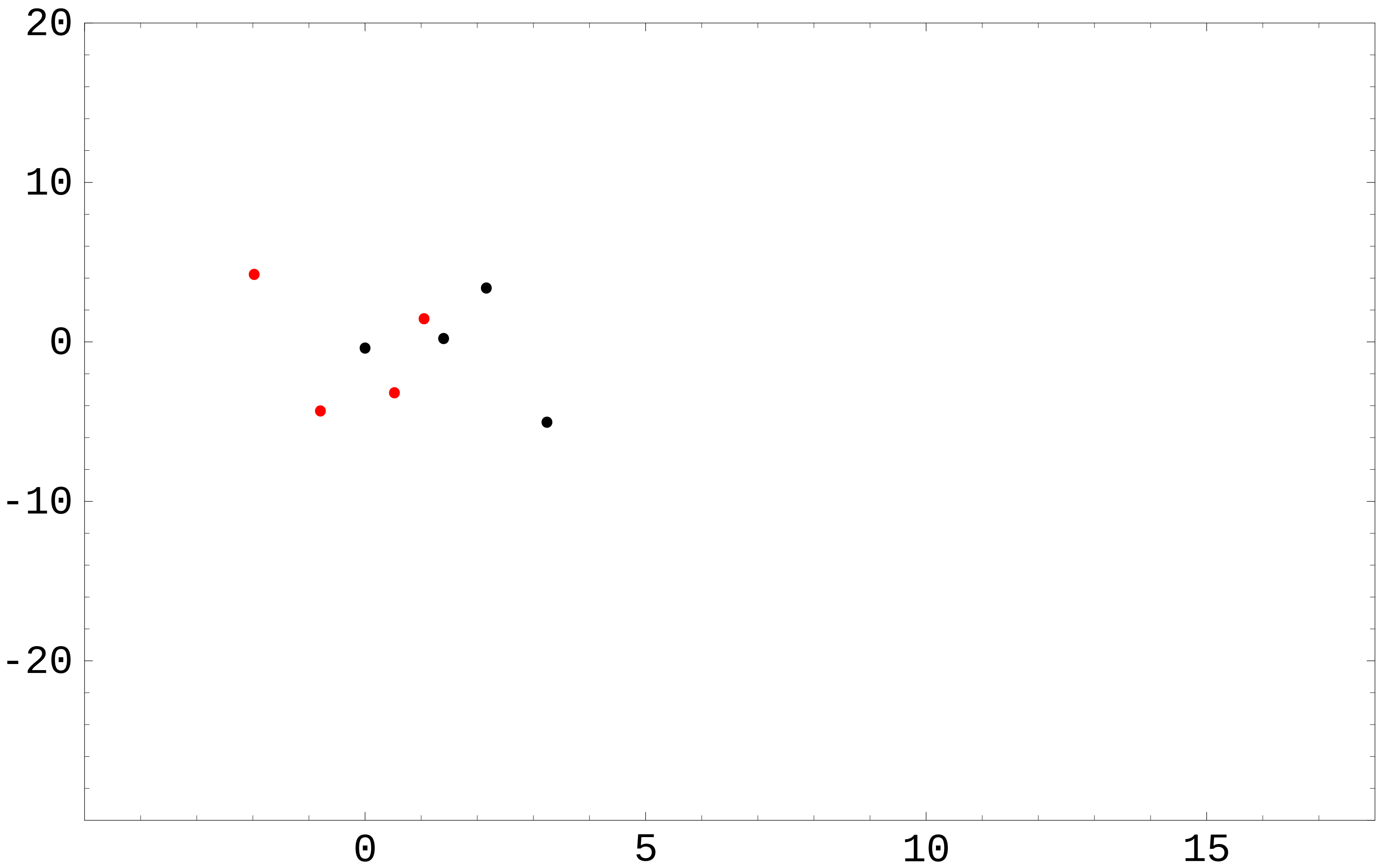
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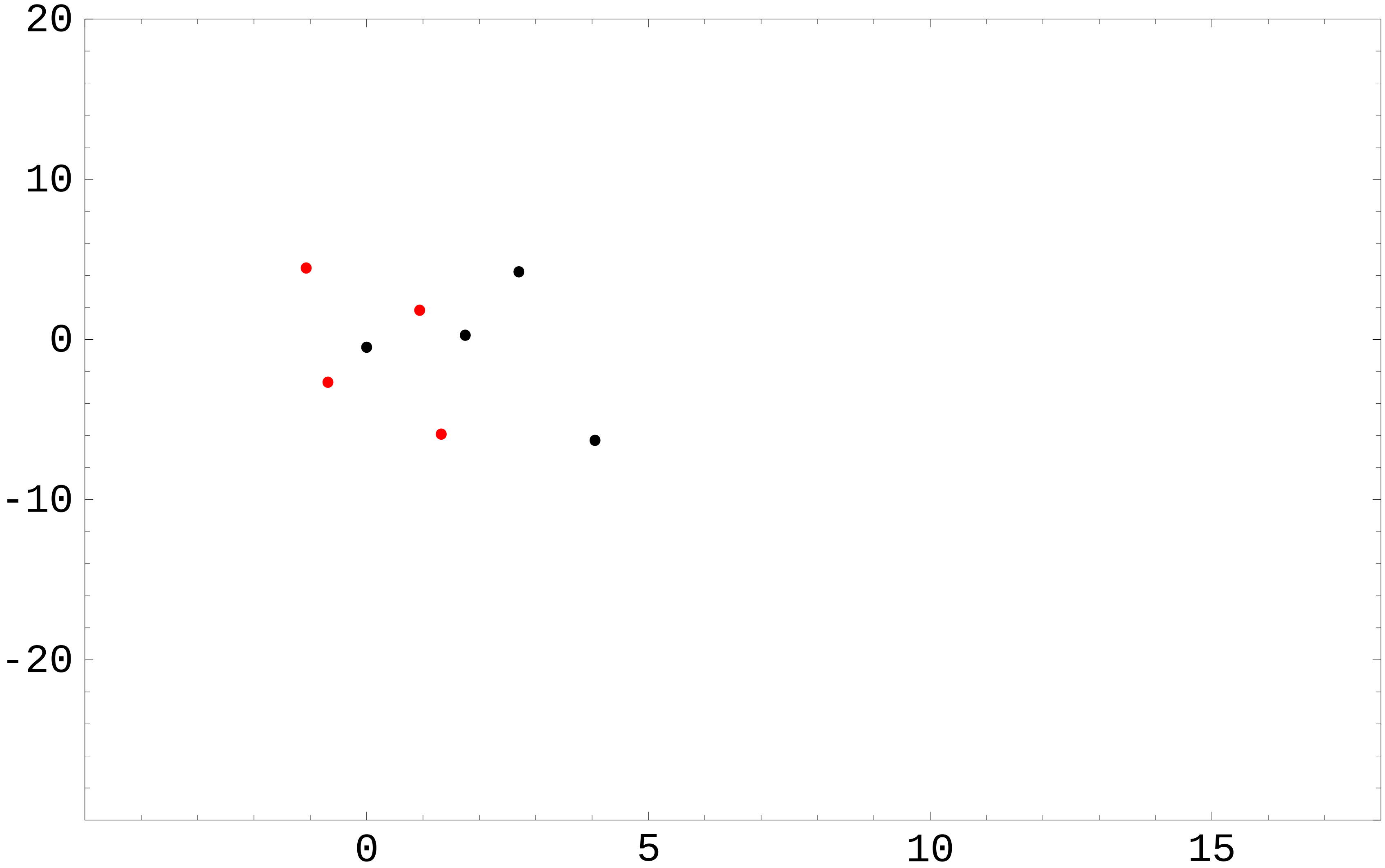
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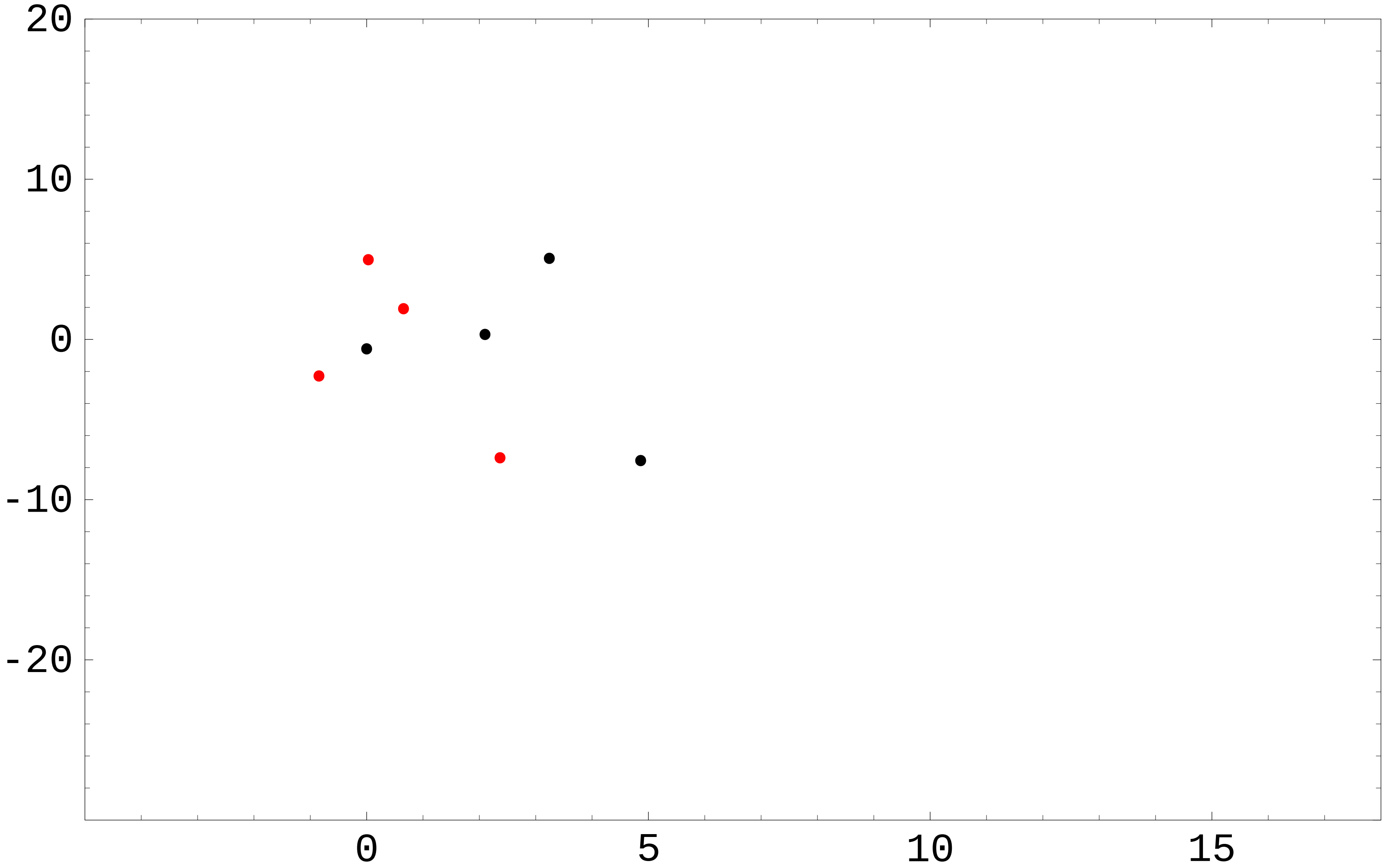

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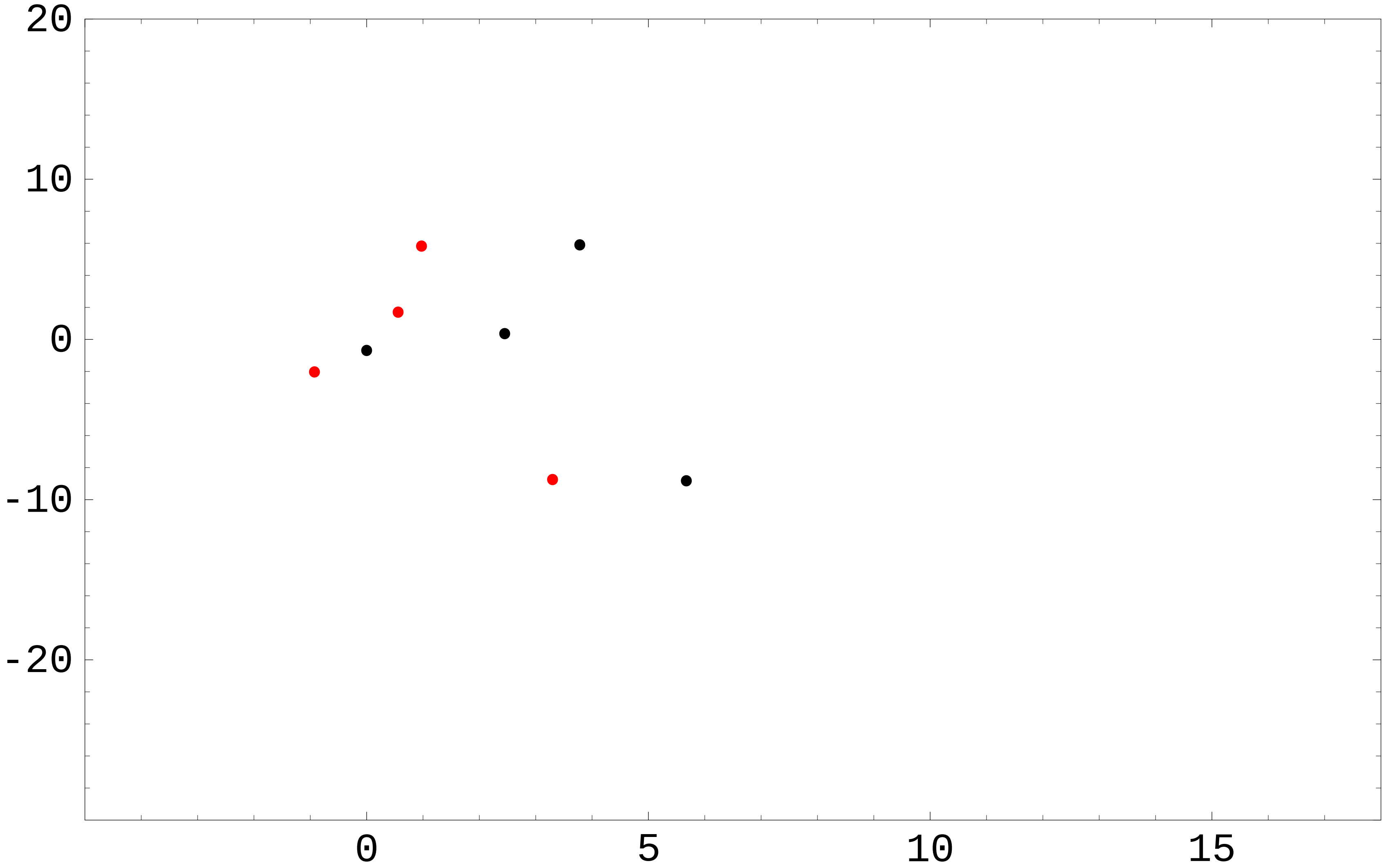
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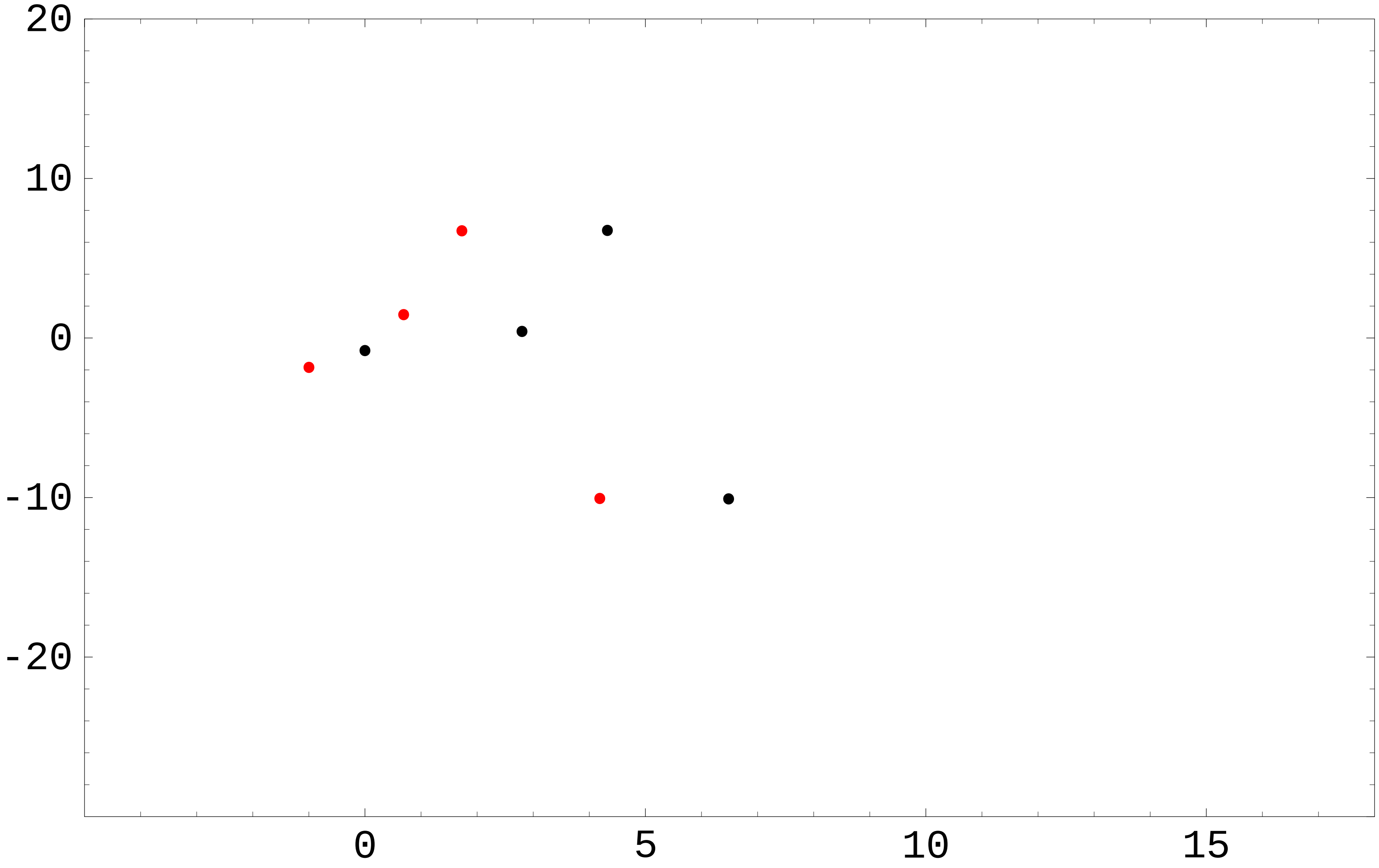
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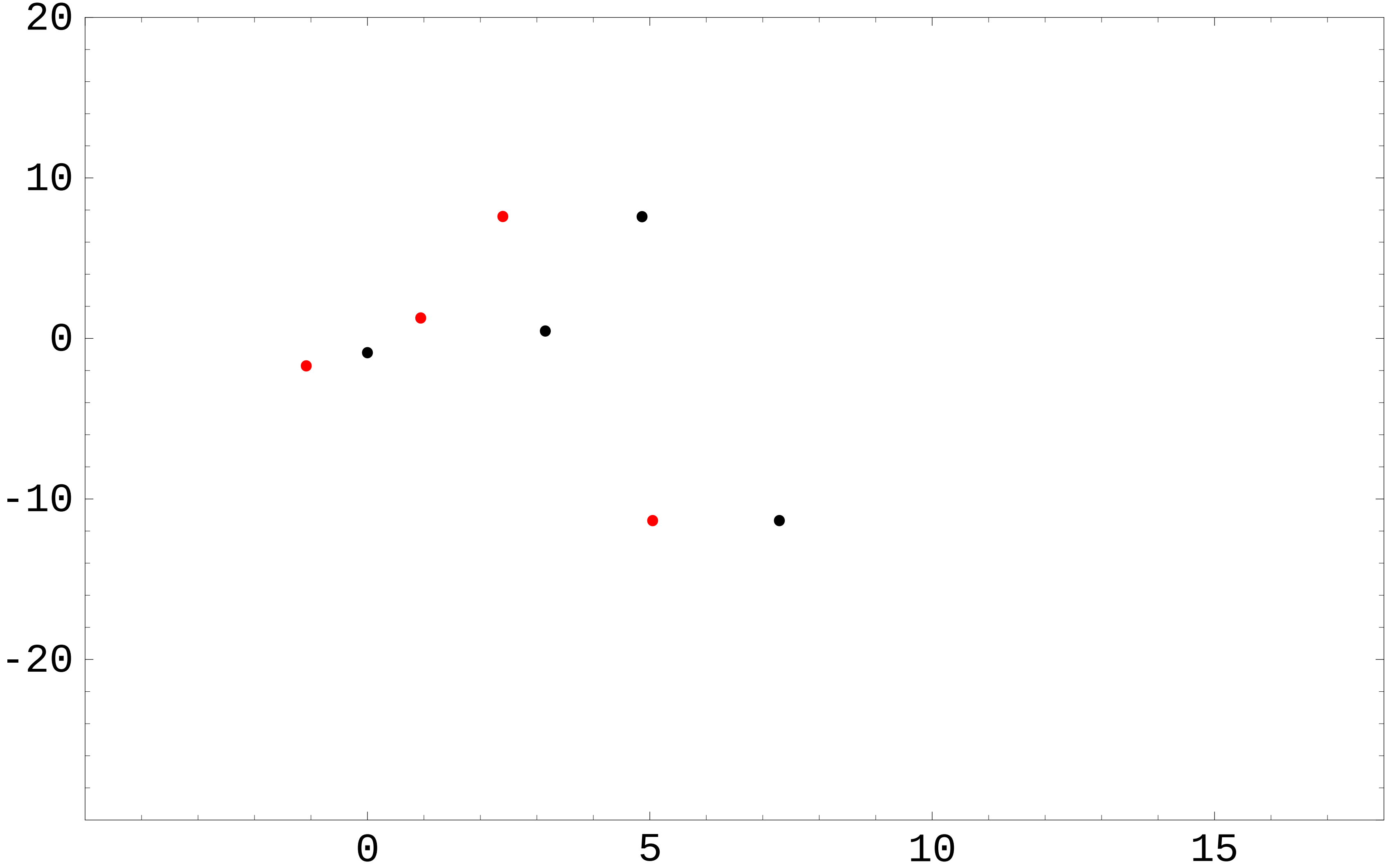

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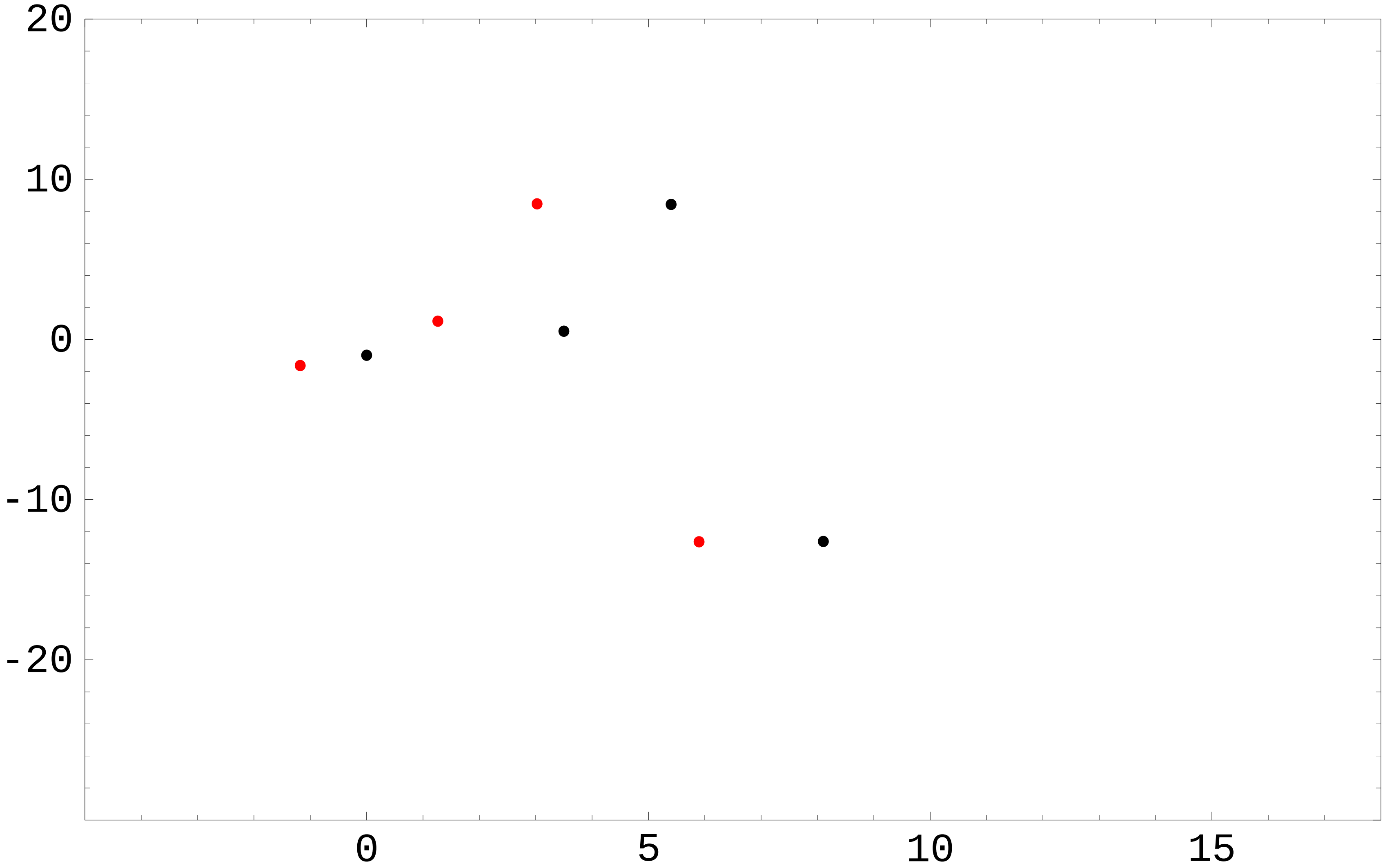
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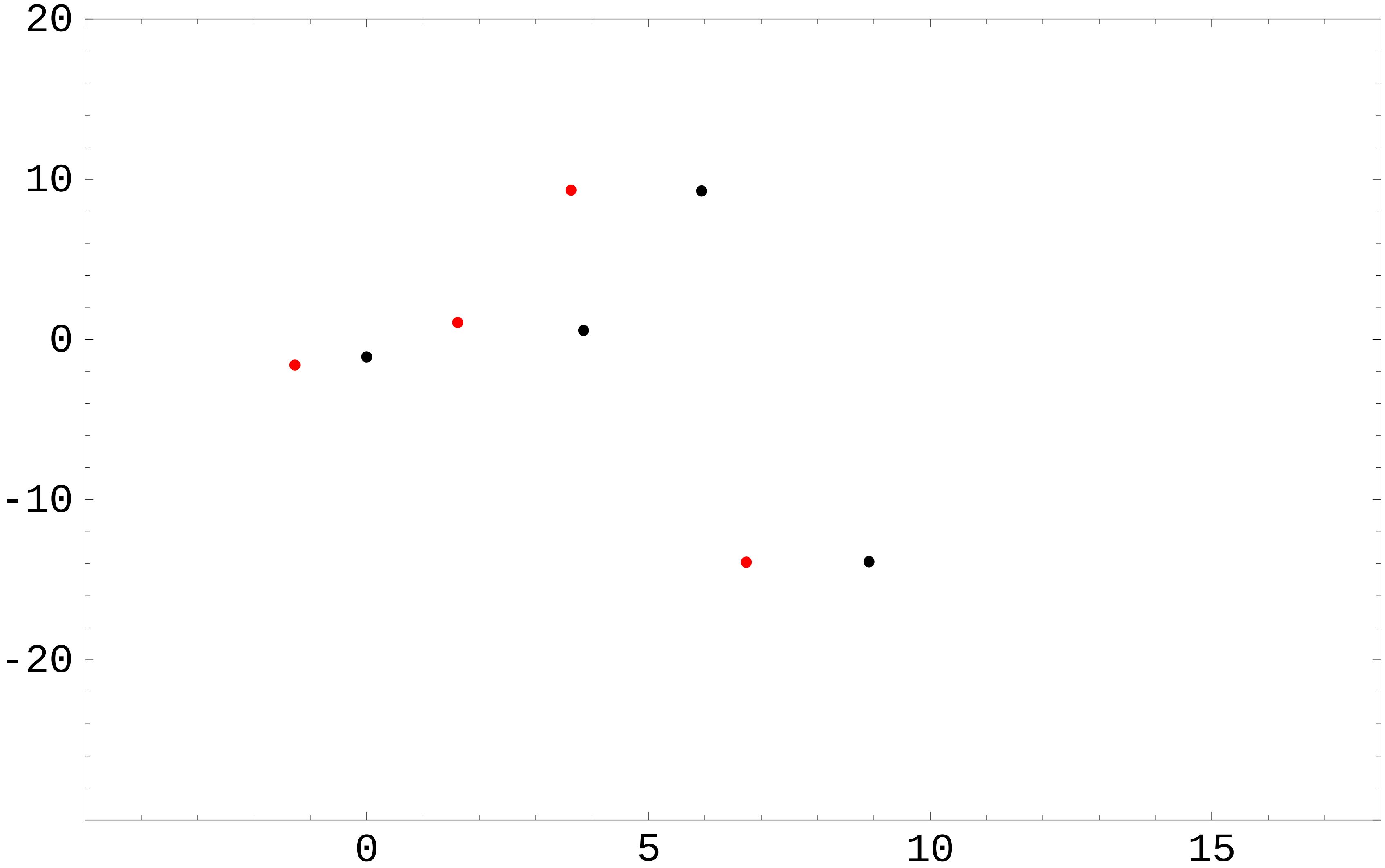
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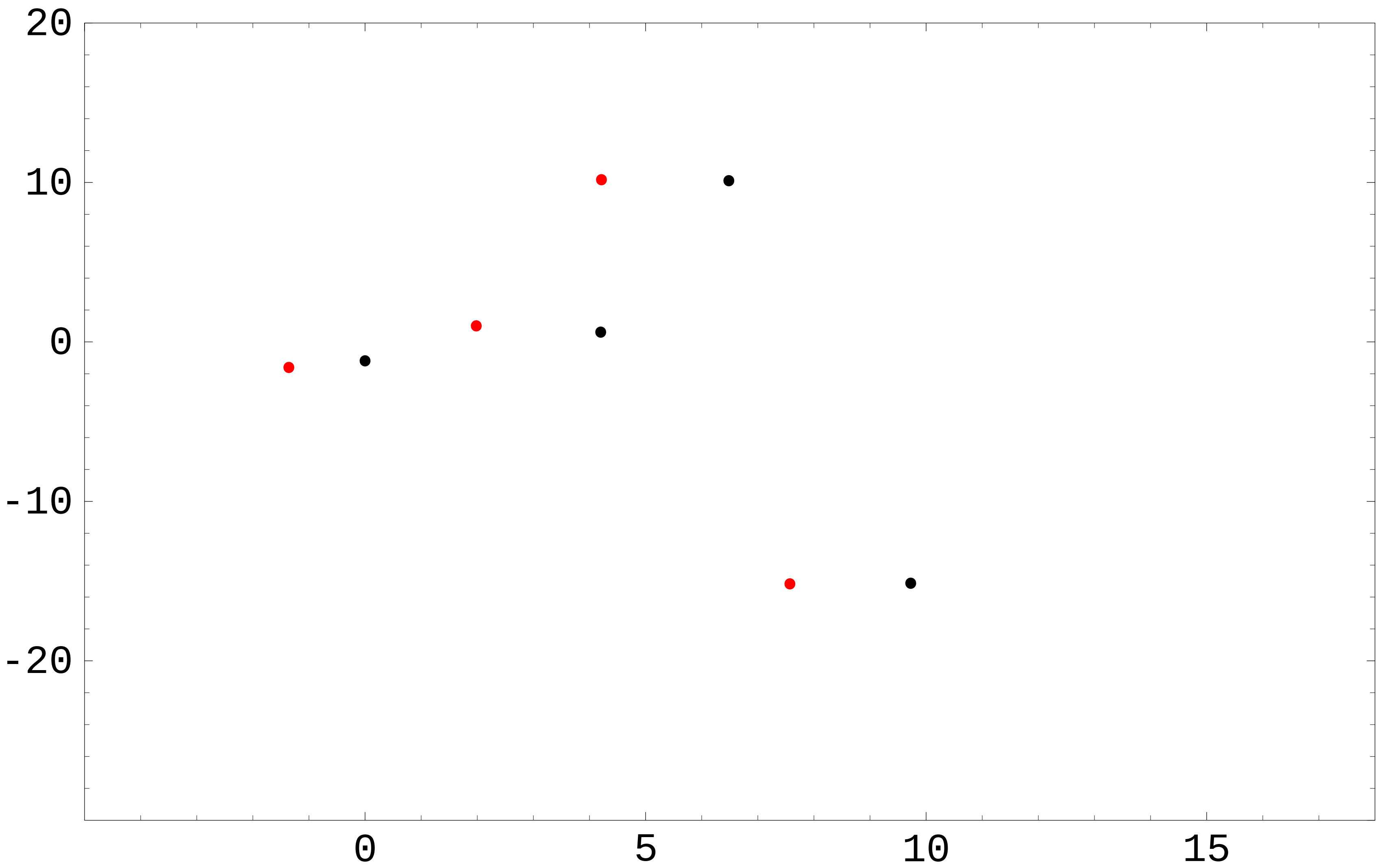
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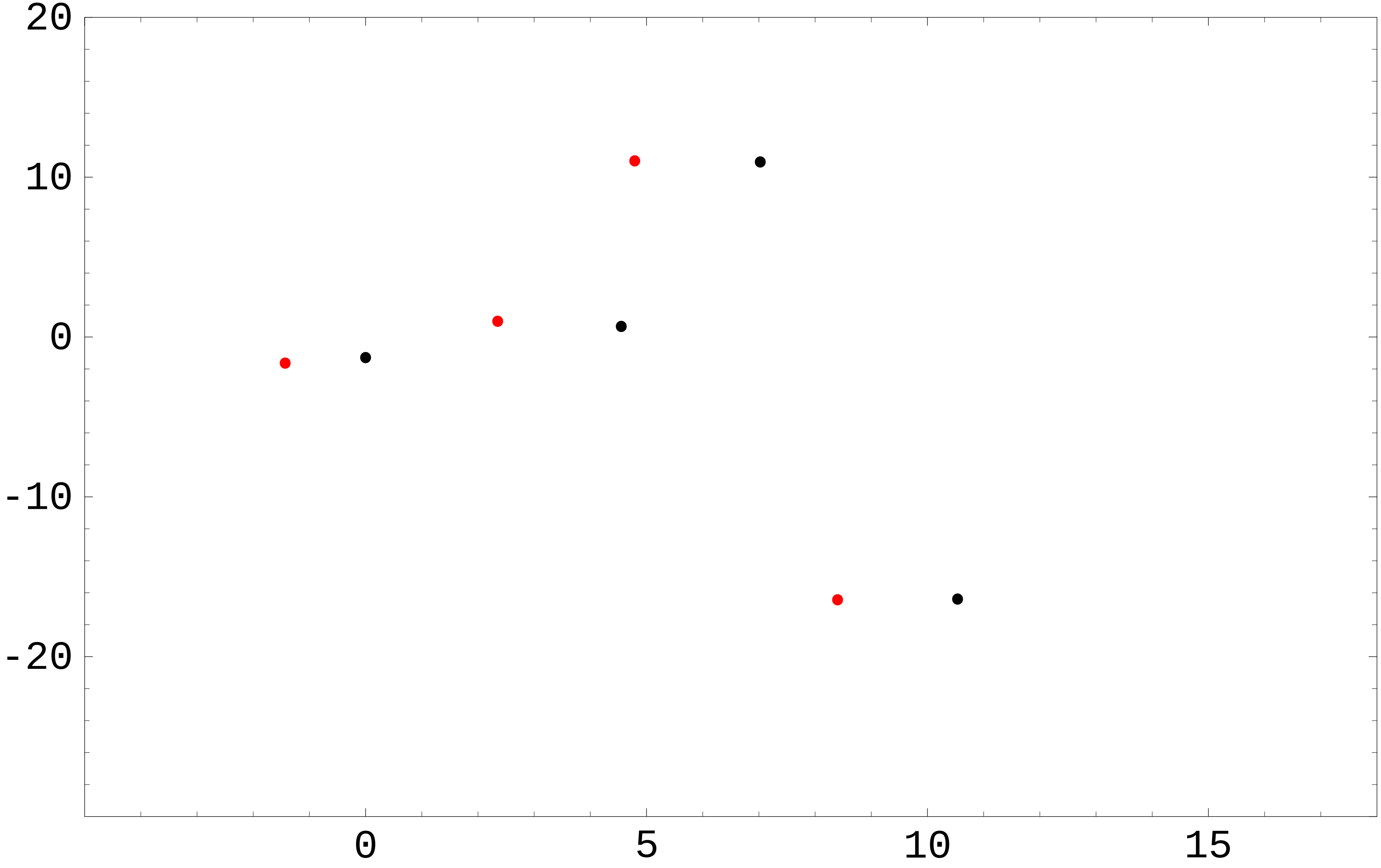
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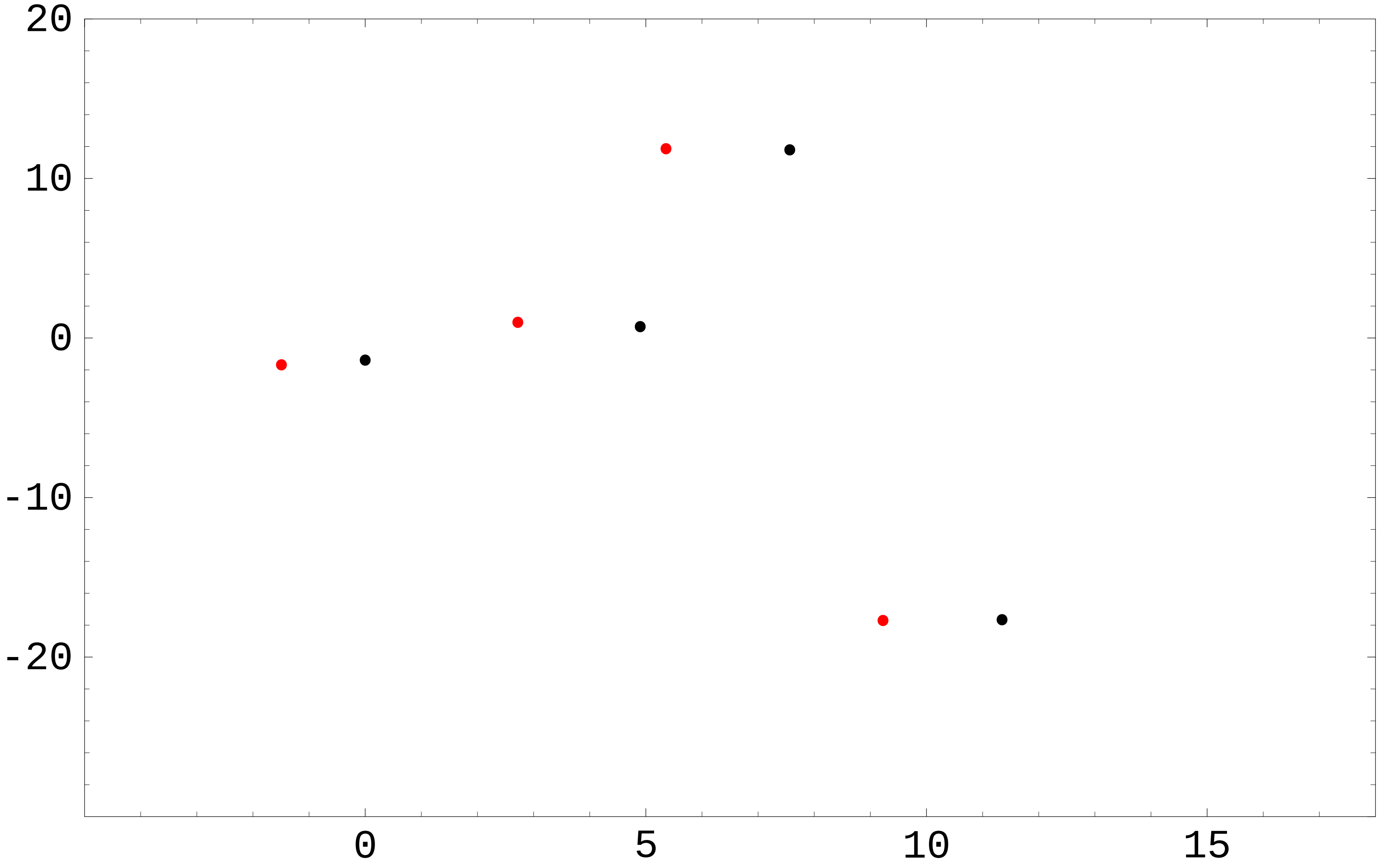
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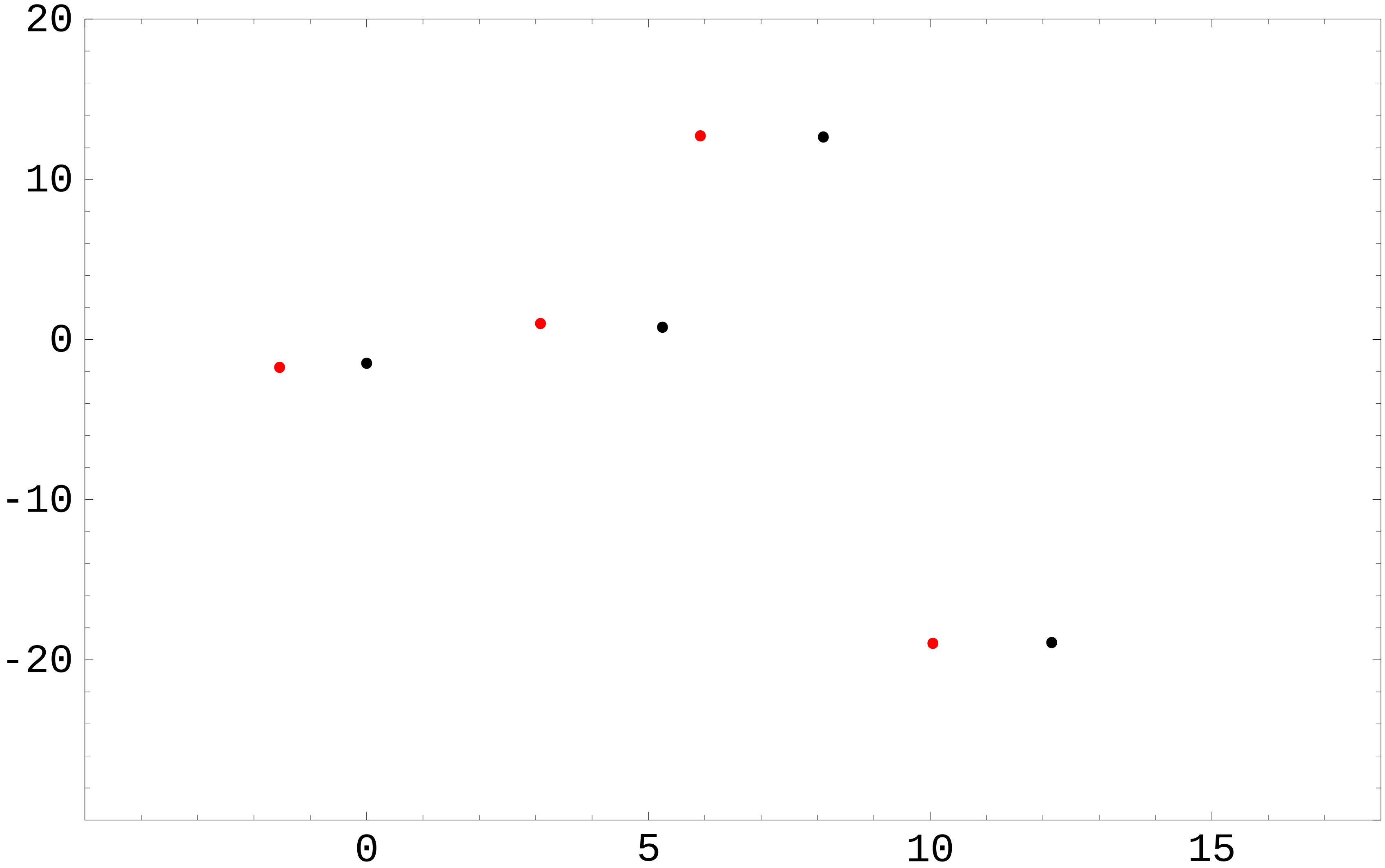

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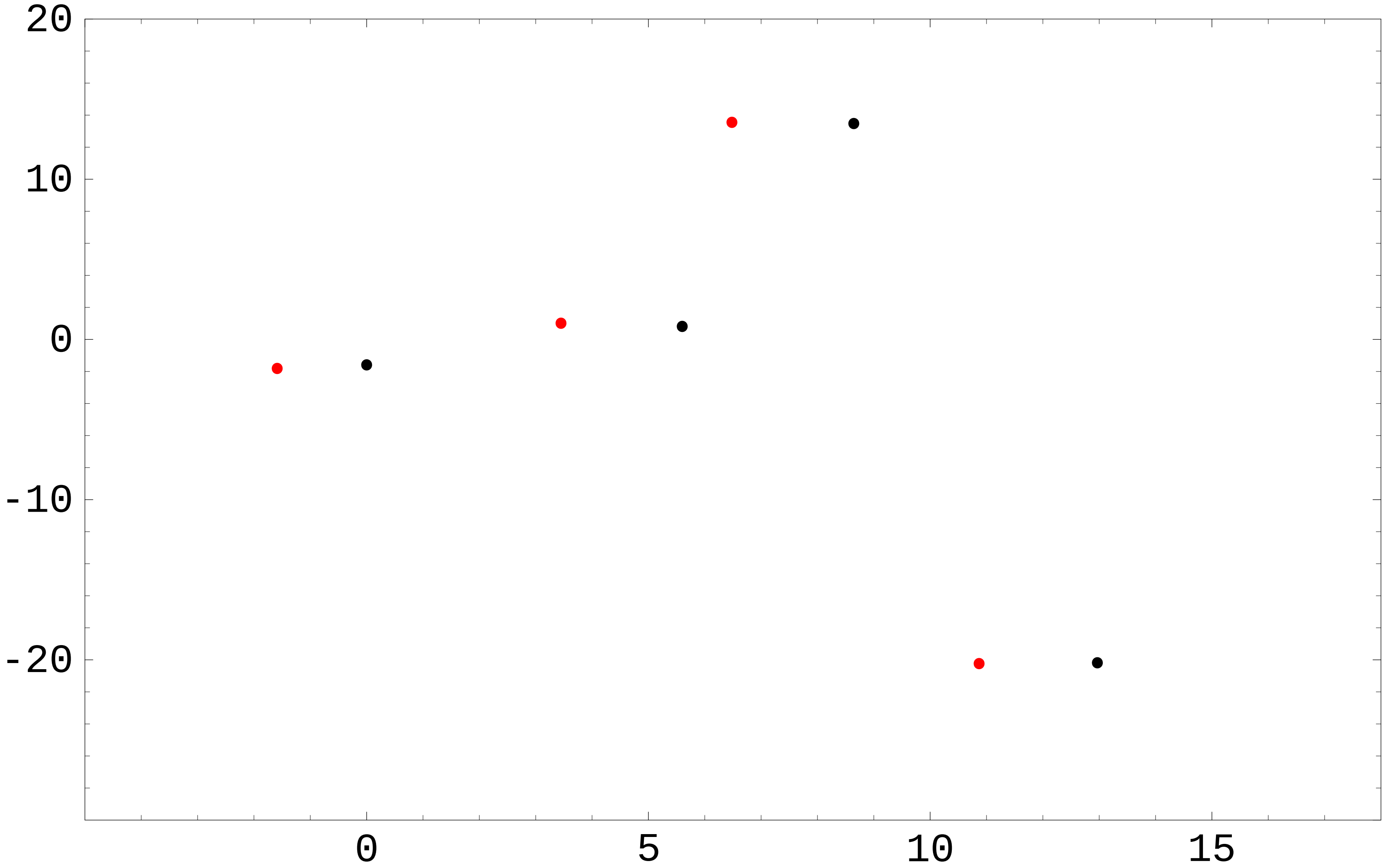

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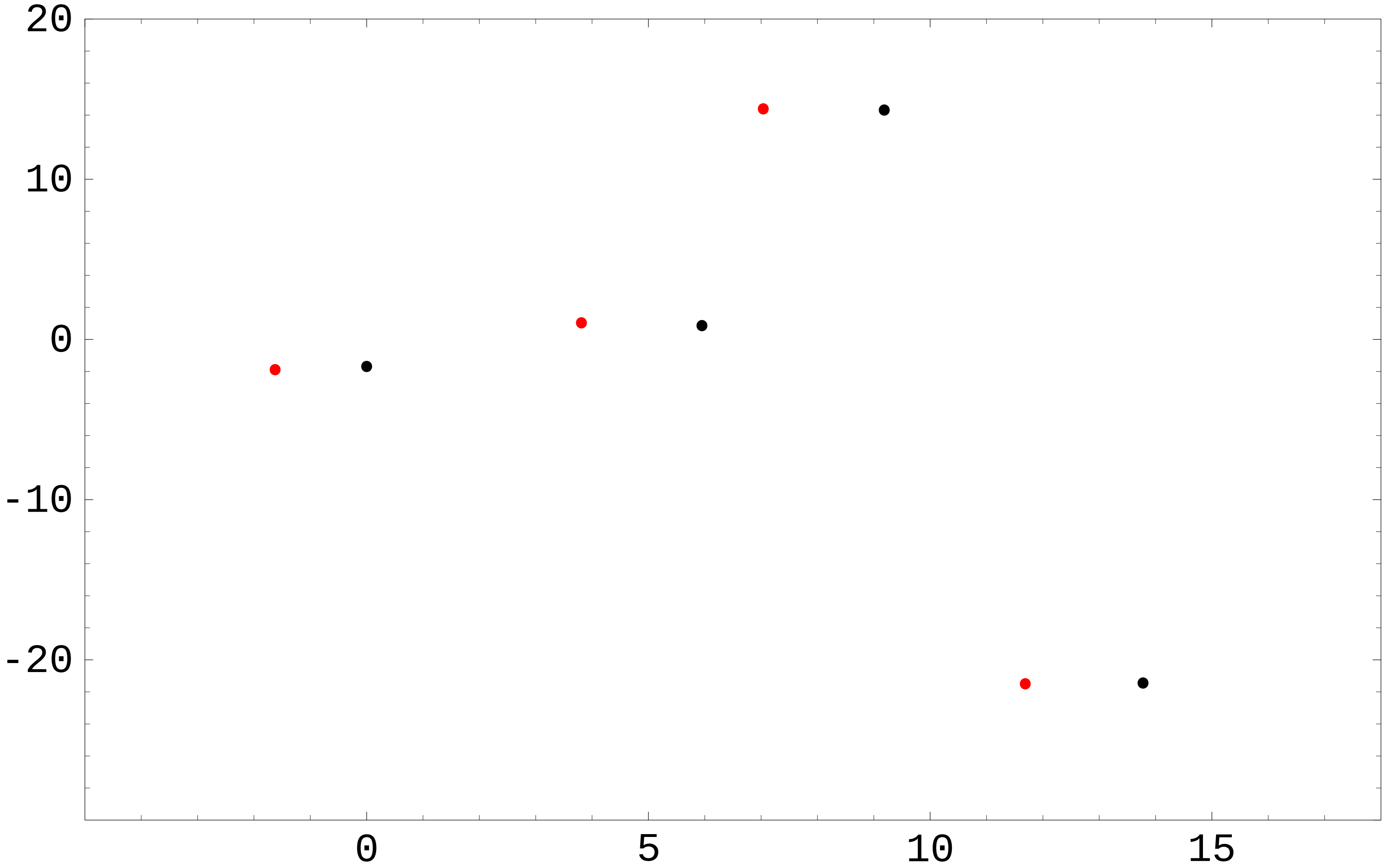
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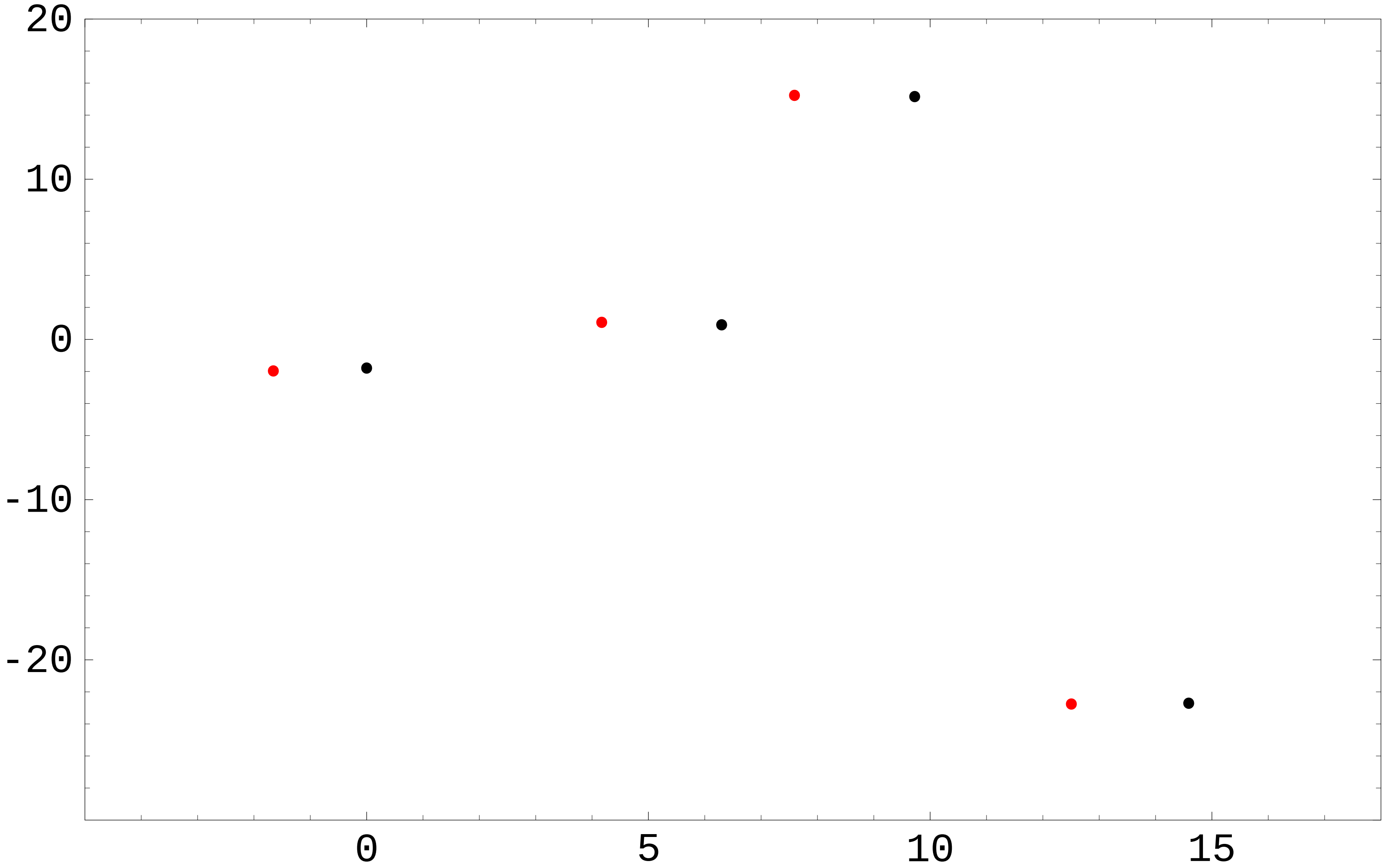
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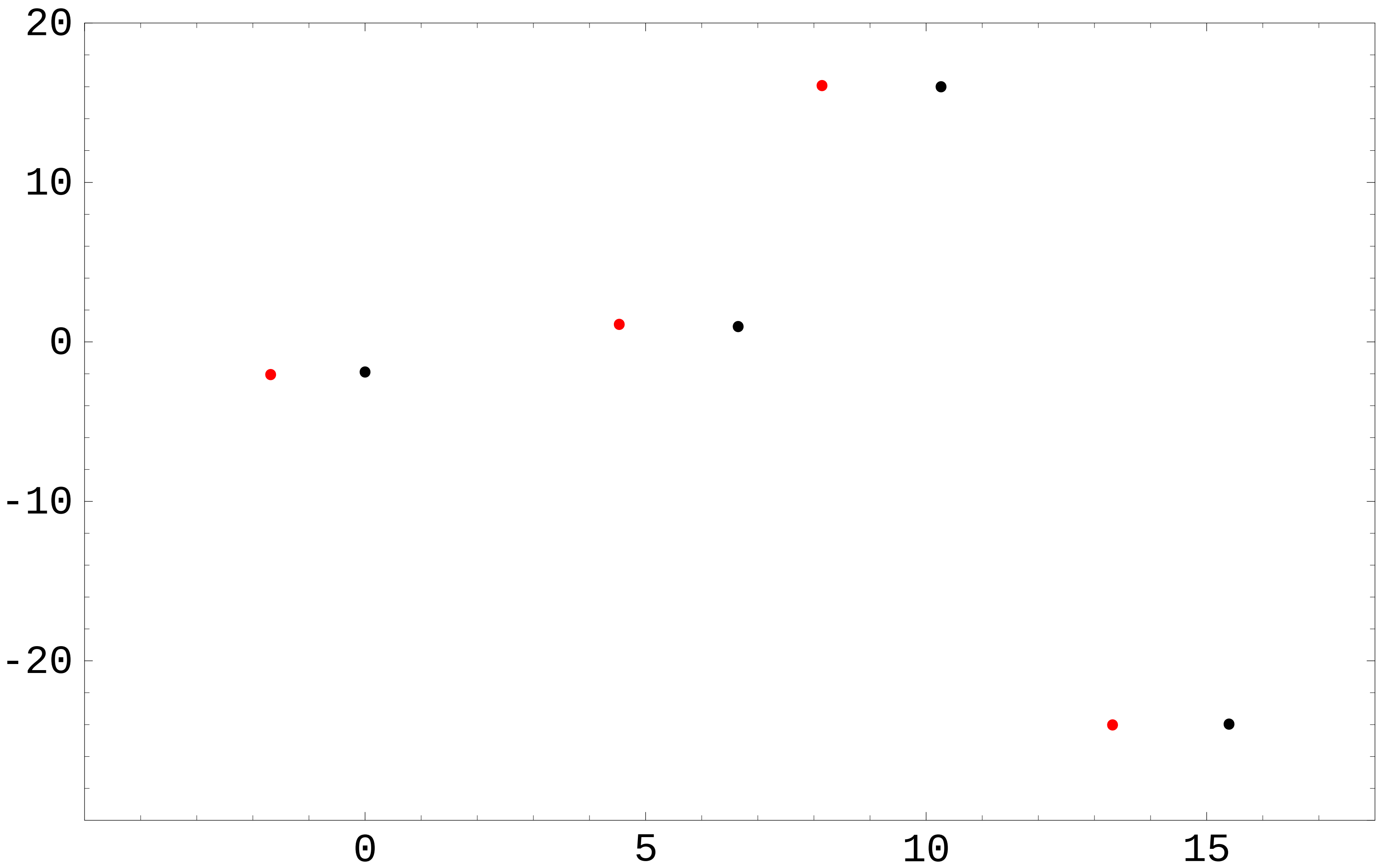
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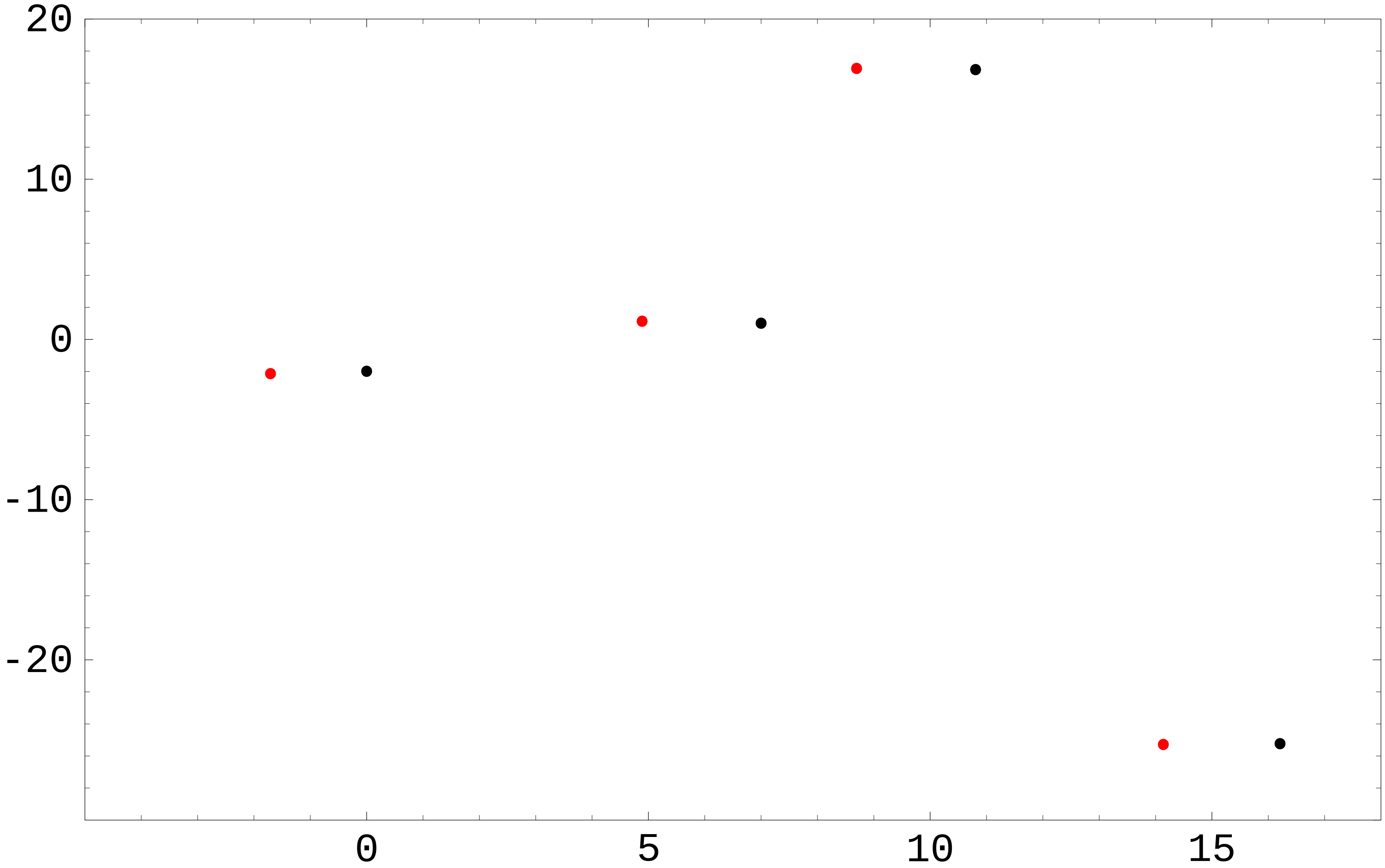
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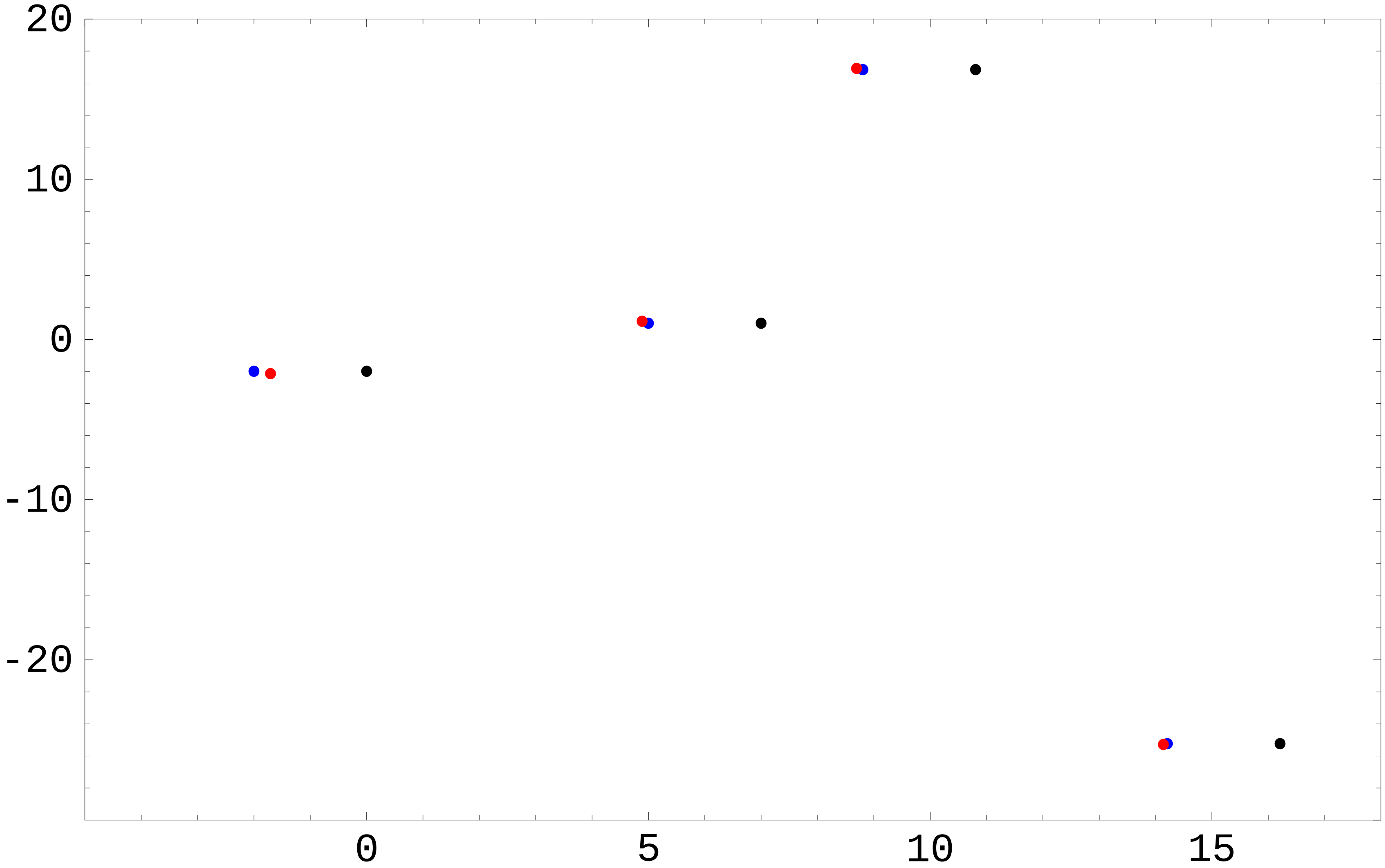

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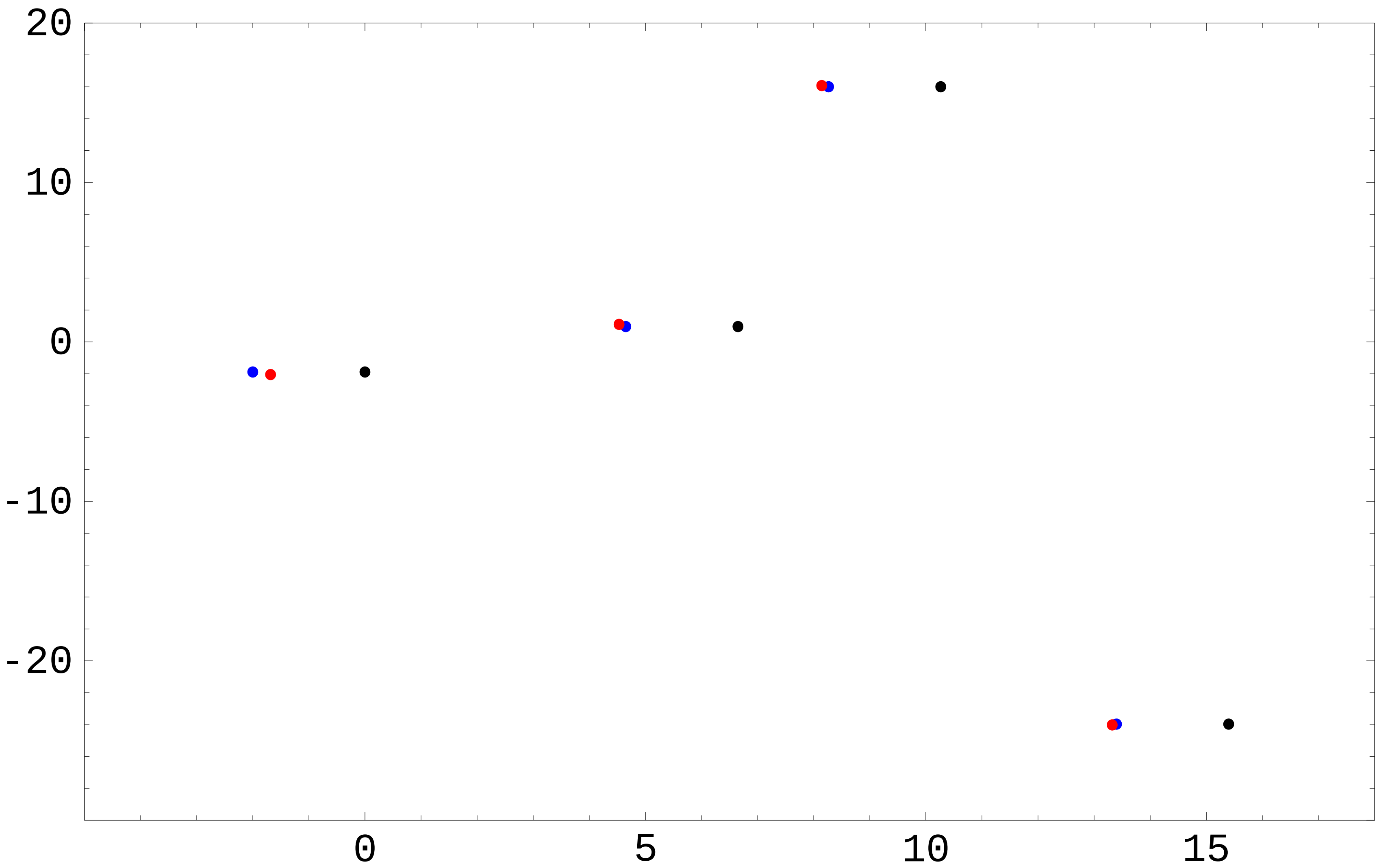
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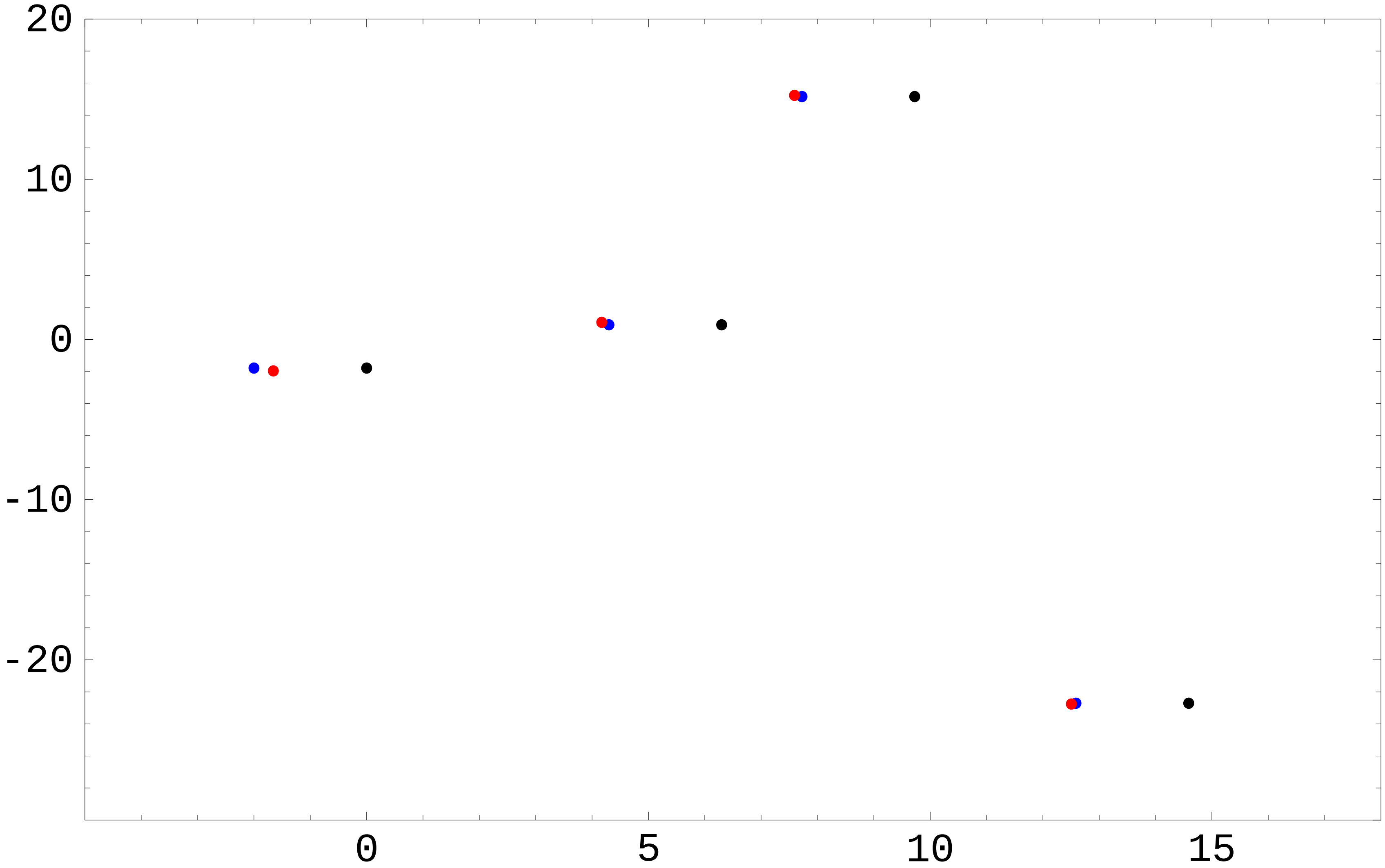
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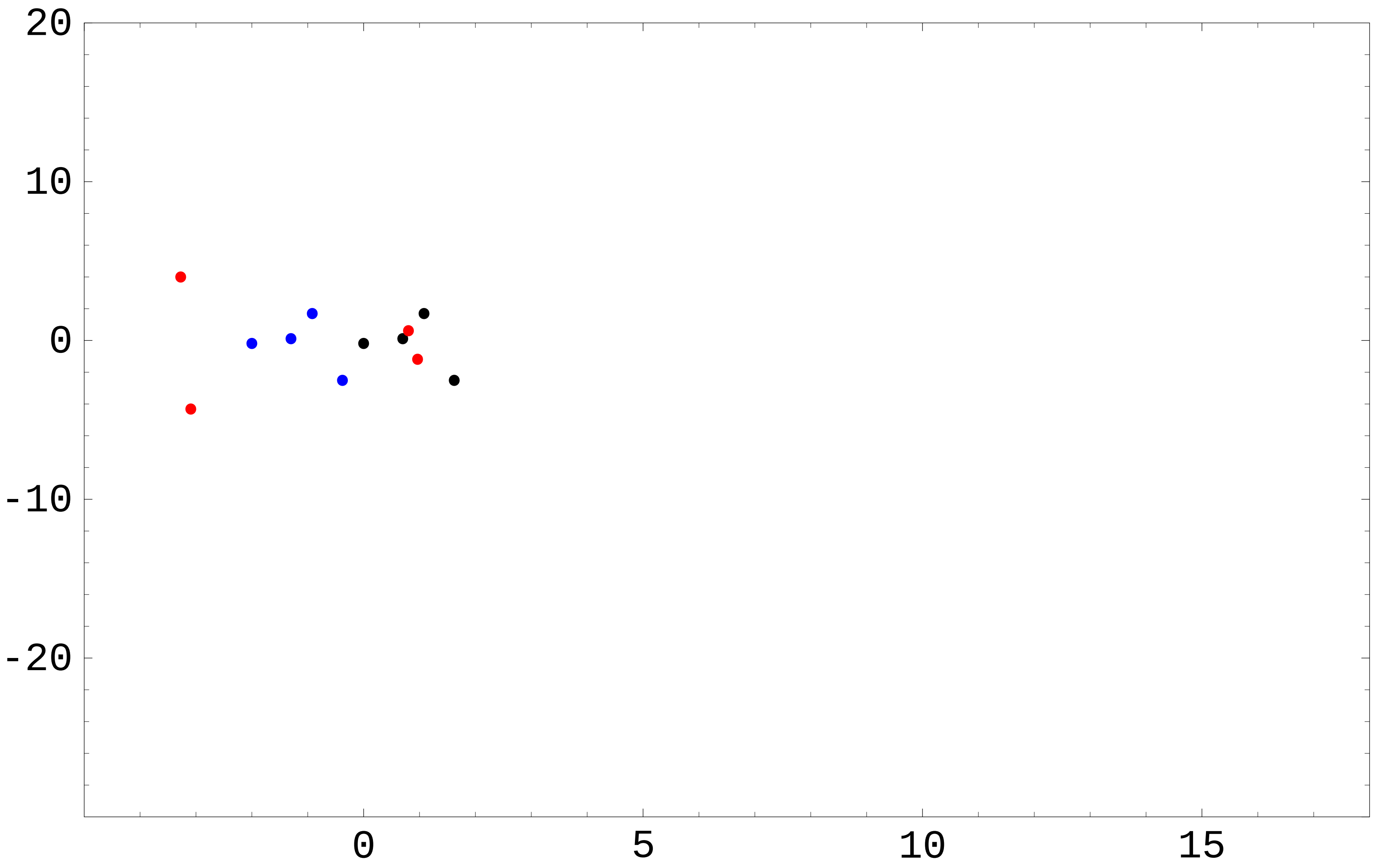
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Let $p \in \mathcal{P}_n$ be a polynomial with n distinct roots.

Then $Z(p) \cap Z(p') = \emptyset$.

Define

$$\tau(p) = \text{min}\Big\{|w - v| : w \in Z(p),\, v \in Z(p')\Big\}$$

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$$\boxed{\mathrm{spr}(p) = \tau(p) \left(\frac{\tau(p)}{\rho(p)}\right)^{n-2}}$$

A trivial example: Let $r > 0$.

$$p(z) = z^n - r^n, \qquad p'(z) = nz^{n-1}$$

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$$\boxed{\mathrm{spr}(p) = r\left(\frac{r}{r}\right)^{n-2} = r}$$

A generalization: Let $t > 0$.

Define $H_t \in \mathcal{L}(\mathcal{P}_n)$ by

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Then

$$\mathsf{spr}(H_t\, p) = t\ \mathsf{spr}(p).$$

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Let $\quad T \ = I + \alpha D + \alpha_2 D^2 + \cdots + \alpha_n D^n$.

Then there exists a constant $\Gamma_T > 0$ such that

$$\boxed{d_F\Big(Z(S_\alpha p), Z(Tp)\Big) \le \frac{\Gamma_T}{\mathrm{spr}(p)}}$$

for all $p \in \mathcal{P}_n$ with n distinct roots.

Recall $S_\alpha = I + \alpha D + \frac{\alpha^2}{2!} D^2 + \cdots + \frac{\alpha^n}{n!} D^n.$

Let $T = I + \alpha D + \alpha_2 D^2 + \cdots + \alpha_n D^n.$

Recall $S_\alpha = I + \alpha D + \frac{\alpha^2}{2!} D^2 + \cdots + \frac{\alpha^n}{n!} D^n.$

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A corollary:

Recall $S_\alpha = I + \alpha D + \frac{\alpha^2}{2!} D^2 + \cdots + \frac{\alpha^n}{n!} D^n$.

Let $\quad T \ = I + \alpha D + \alpha_2 D^2 + \cdots + \alpha_n D^n$.

A corollary:

For an arbitrary $p \in \mathcal{P}_n$ with n distinct roots:

$$\boxed{\lim_{t \to +\infty} d_F\Big(Z(S_\alpha H_t p), Z(T H_t p) \Big) = 0}$$